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# LINEAR SOLUTIONS OF THE GEODETIC BOUNDARY-VALUE PROBLEM

by  
Helmut Moritz

Prepared for  
Air Force Cambridge Research Laboratories  
Office of Aerospace Research  
United States Air Force  
Bedford, Massachusetts 01730

Contract No. AF19(628)-5701  
Project No. 7600  
Task No. 760002,04

Scientific Report No. 11



The Ohio State University  
Research Foundation  
Columbus, Ohio 43212

December, 1966



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## FOREWORD

This report was prepared by Helmut Moritz, Professor, Technische Universität Berlin and Research Associate, Department of Geodetic Science of The Ohio State University, under Air Force Contract No. AF19(628)-5701, OSURF Project No. 2122, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen W. Williams and Mr. Bela Szabo, Project Scientists.

## ABSTRACT

This report is concerned with formulas for the determination of the earth's physical surface and external gravity field from free-air gravity anomalies to an approximation linear in the elevation and its derivatives.

Part A considers integral equations and their linear solutions; Part B gives an elementary deduction of these solutions from the geometrically evident gradient solution; and the subject of Part C is an application to various gravity-dependent quantities and an evaluation of different solutions.

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## INTRODUCTION

The present work is a continuation of an earlier report (Moritz, 1964).

The previous report gave a discussion of the various formulations of the geodetic boundary-value problem, that is, the gravimetric determination of the geoid or of the physical surface of the earth, using either unreduced free-air anomalies or other anomalies that correspond to various gravity reductions. The present report is exclusively concerned with "Molodensky's problem," the gravimetric determination of the earth's physical surface and external gravity field from free-air anomalies. The free-air anomaly  $\Delta g$  is defined as the difference between gravity  $g$  measured at ground and normal gravity  $\gamma$  referred to the telluroid. The discussion of the solutions of Molodensky's problem is in general limited to "linear solutions," in which second and higher powers of the elevation (which is small as compared to the dimensions of the earth) and of the terrain inclination are systematically neglected. This limitation to a "linear approximation" considerably simplifies the comparative study of the different solutions and is also justified for practical reasons.

In Molodensky's problem the free-air anomalies are assumed to be given at every point of the earth's surface. This distinguishes it from "Bjerhammar's problem," the determination of the earth's physical surface and external gravity field from discrete gravity measurements, which is beyond the scope of the present report. We have also disregarded methods, mainly proposed by N. K. Migal,

that avoid the use of a normal gravity field: they are interesting conceptually, but less convenient practically and, as shown by Monin (1962), essentially equivalent to the usual methods, which do use a normal field.

Part A considers various integral equations for the present problem, and their linear solutions. Much of its subject matter can be found in the literature; it has been collected, supplemented, and presented from a unified point of view.

Part B gives a deduction of the linear solutions from one particularly simple and obvious solution. Here I have attempted to show that all linear solutions that I was able to find in the literature, and some more, can be derived in an elementary way, in the sense that no integral equations are needed at all. In this way the relation between the various, apparently so different, solutions is clarified. This comprehensive deduction may claim some originality, although the general technique was anticipated, for a particular case, by Arnold and indirectly also by Molodensky.

Part C considers the application of the various methods of solution to gravity-dependent quantities that are needed in geodesy. A comparison and evaluation, both theoretically and with respect to practical application, is attempted.

Mathematical details that would unduly interrupt the presentation have been relegated to two Appendices.



Formulas that might be new are, for instance, (63), (106), (261), (274), (277), and (297b). It is quite possible, however, that some of them have been found before.

The three parts of the report can be read fairly independently. Readers interested only in a general view and in practical application may limit themselves to Part C; and those who are afraid of integral equations may start with Part B, although they will probably have to refer to sections 1 and 3 for a full understanding.

## A. THE INTEGRAL EQUATIONS

### 1. SPHERICAL FORMULAS

For later reference we first state certain well-known integral formulas, which are valid if the boundary surface is a sphere.

SPHERICAL HARMONICS EQUIVALENTS. Let  $r$  (radius vector),  $\theta$  (polar distance) and  $\lambda$  (longitude) be spherical polar coordinates in space. Then  $r = R = 6371$  km is the radius of the mean terrestrial sphere, considered as the boundary surface. Consider on this sphere a surface layer of density  $X(\theta, \lambda)/2\pi k$ , where  $k$  is the gravitational constant, and its potential  $Y(\theta, \lambda)$  on the sphere. These two functions are connected by the formula

$$Y = \frac{R^2}{2\pi} \iint_{\sigma} \frac{X}{l_0} d\sigma, \quad (1)$$

where  $Y$  refers to a fixed point  $P$  and  $X$  to a variable point  $P'$  that carries the surface element  $R^2 d\sigma$ ,  $\sigma$  being the concentric unit sphere;  $l_0$  is the spatial distance and  $\psi$  the spherical distance between  $P$  and  $P'$ , so that

$$l_0 = 2R \sin \frac{\psi}{2} \quad (2)$$

(Fig. 1).

Representing the functions  $X$  and  $Y$  as series of spherical harmonics,

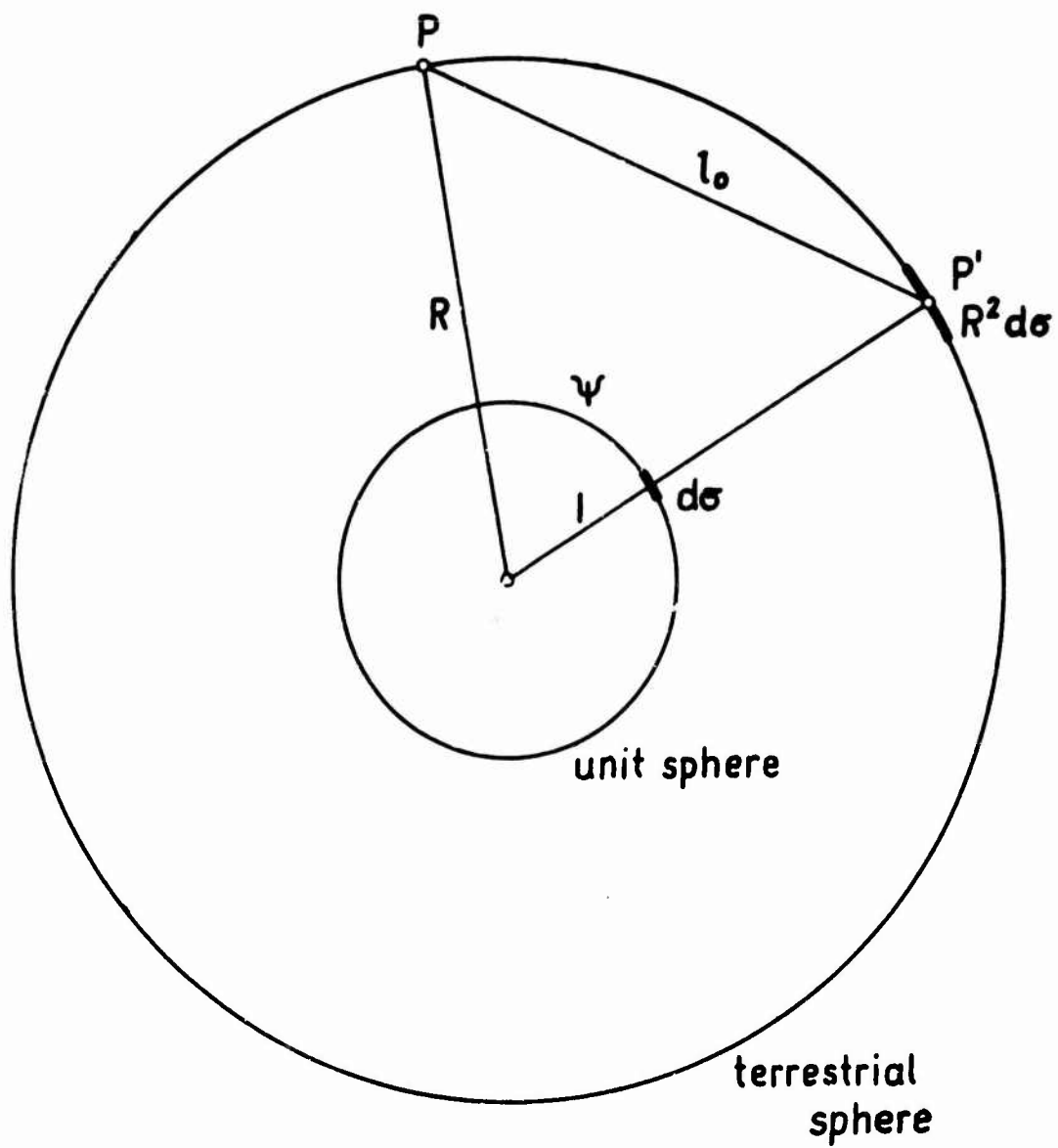


Figure 1

$$X(\theta, \lambda) = \sum_{n=0}^{\infty} X_n(\theta, \lambda), \quad (3)$$

$$Y(\theta, \lambda) = \sum_{n=0}^{\infty} Y_n(\theta, \lambda),$$

equation (1) is equivalent to the relation

$$Y_n(\theta, \lambda) = \frac{2R}{2n+1} X_n(\theta, \lambda) \quad (4)$$

between the harmonic terms of degree  $n$ .

Similarly Stokes' formula

$$Y = \frac{R}{4\pi} \iint_{\sigma} U S(\psi) d\sigma \quad (5)$$

( $Y$  represents the anomalous potential and  $U$  the gravity anomaly) is equivalent to

$$Y_n(\theta, \lambda) = \frac{R}{n-1} U_n(\theta, \lambda). \quad (6)$$

To obtain the formulas inverse to (1) and (5), we consider the elementary inversion of the corresponding relations (4) and (6):

$$R X_n = (n + \frac{1}{2}) Y_n,$$

$$R U_n = (n - 1) Y_n$$

or

$$(R X - \frac{1}{2} Y)_n = n Y_n, \quad (7)$$

$$(R U + Y)_n = n Y_n.$$

Both equations are of the form

$$Z_n = n Y_n, \quad (8)$$

which is the spherical harmonics equivalent of the integral formula

$$Z = -\frac{R^3}{2\pi} \iint_{\sigma} \frac{Y - Y_p}{l_0^3} d\sigma \quad (9)$$

(Heiskanen and Moritz, 1967, sec. 1-18). Outside the integral, functions (such as  $Z$ ) always refer to the fixed point  $P$  under consideration; inside the integral, functions (such as  $Y$ ) are in general variables of integration, and therefore the value of this function at  $P$  is then explicitly denoted by the subscript  $P$  (such as  $Y_p$ ).

Now the inversion of (1) and (5) is straightforward. By means of (7), (8), and (9) we obtain

$$X = \frac{Y}{2R} - \frac{R^2}{2\pi} \iint_{\sigma} \frac{Y - Y_p}{l_c^3} d\sigma, \quad (10)$$

$$U = -\frac{Y}{R} - \frac{R^2}{2\pi} \iint_{\sigma} \frac{Y - Y_p}{l_0^3} d\sigma. \quad (11)$$

#### APPLICATION TO THE ANOMALOUS GRAVITY FIELD. The spherical-

harmonic expansion of the anomalous potential in space may be written

$$T = \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+1} T_n, \quad (12)$$

first and second degree terms being omitted as usually. Differentiating we find

$$-\frac{\partial T}{\partial r} = \sum_{n=2}^{\infty} (n+1) \frac{R^{n+1}}{r^{n+2}} T_n.$$

The gravity anomaly in space is given by

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2T}{r} = \sum_{n=2}^{\infty} (n-1) \frac{R^{n+1}}{r^{n+2}} T_n.$$

We write this as

$$\Delta g = \sum_{n=2}^{\infty} \left( \frac{R}{r} \right)^{n+2} \Delta g_n$$

and differentiate with respect to  $r$ :

$$\frac{\partial \Delta g}{\partial r} = - \sum_{n=2}^{\infty} (n+2) \frac{R^{n+2}}{r^{n+3}} \Delta g_n .$$

By setting  $r = R$  we specialize these formulas for the mean terrestrial sphere which approximates the earth's surface:

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n = \sum_{n=2}^{\infty} \frac{n-1}{R} T_n , \quad (13)$$

$$\frac{\partial \Delta g}{\partial r} = - \frac{1}{R} \sum_{n=2}^{\infty} (n+2) \Delta g_n . \quad (14)$$

The corresponding integral formulas are

$$\Delta g = - \frac{T}{R} - \frac{R^2}{2\pi} \iint_{\sigma} \frac{T - T_p}{l_o^3} d\sigma , \quad (15)$$

$$\frac{\partial \Delta g}{\partial r} = - \frac{2\Delta g}{R} + \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g - \Delta g_p}{l_o^3} d\sigma . \quad (16)$$

The first is identical with (11), the second is derived in the same way, using (14), (8), and (9).

Equation (16) expresses the vertical gradient of the gravity anomaly in terms of the gravity anomaly itself; equation (15) is the inverse of Stokes' formula.

INTEGRAL EQUATIONS. By applying Green's identities to a spherical boundary surface one obtains the integral equation

$$T - \frac{3R}{4\pi} \iint_{\sigma} \frac{T}{l_o} d\sigma = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g}{l_o} d\sigma ; \quad (17)$$

see, e.g., (Moritz, 1964). Its solution is Stokes' formula (5):

$$T = \frac{R}{4\pi} \iint_{\sigma} \Delta g S(\psi) d\sigma. \quad (18)$$

Another integral equation is obtained by representing  $T$  as the potential of a surface layer. Denote the product of surface density and gravitational constant by  $\varphi$ ; then

$$T = R^2 \iint_{\sigma} \frac{\varphi}{\ell_0} d\sigma. \quad (19)$$

Inserting this into the boundary condition

$$\frac{\partial T}{\partial r} + \frac{2T}{R} + \Delta g = 0 \quad (20)$$

and taking account of the discontinuity of the normal derivative of a surface potential we obtain the integral equation for  $\varphi$ ,

$$\varphi - \frac{3R}{4\pi} \iint_{\sigma} \frac{\varphi}{\ell_0} d\sigma = \frac{1}{2\pi} \Delta g; \quad (21)$$

see, e.g., (Heiskanen and Moritz, 1967, sec. 8-6). Inserting (19) into (21) we find

$$\varphi = \frac{1}{2\pi} \left( \Delta g + \frac{3T}{2R} \right). \quad (22)$$

Expressing  $T$  by Stokes' formula (18) we obtain

$$\varphi = \frac{1}{2\pi} \left( \Delta g + \frac{3}{8\pi} \iint_{\sigma} \Delta g S(\psi) d\sigma \right) \quad (23)$$

as a solution of the integral equation (21).

GREEN'S SURFACE IDENTITIES. Let  $U$  and  $V$  be two functions defined

on an arbitrary surface  $S$ . Then the relation

$$\iint_{S_1} \bar{D}(U, V) dS = - \int_C U \frac{\partial V}{\partial \nu} ds - \iint_{S_1} U \Delta_2 V dS \quad (24)$$

holds. Here  $C$  is a closed curve on the surface,  $S_1$  is the part of the surface  $S$  that is enclosed by  $C$ ,  $ds$  is the line element of  $C$ ,  $dS$  is the surface element of  $S$ ,  $\nu$  is the tangent to  $S$  that is normal to  $C$ ,  $\bar{D}$  is Beltrami's mixed differential parameter, and  $\Delta_2$  is Beltrami's second differential parameter for the surface. See (McConnell, 1931), p. 189, eq. (69); the notation is slightly different.

If the surface is referred to orthogonal parameters  $u_1$  and  $u_2$ , so that the line element is represented by

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2, \quad (25)$$

then the differential parameters are

$$\bar{D}(U, V) = \frac{1}{h_1^2} \frac{\partial U}{\partial u_1} \frac{\partial V}{\partial u_1} + \frac{1}{h_2^2} \frac{\partial U}{\partial u_2} \frac{\partial V}{\partial u_2}, \quad (26)$$

$$\Delta_2(F) = \frac{1}{h_1 h_2} \left[ \frac{\partial}{\partial u_1} \left( \frac{h_2}{h_1} \frac{\partial F}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_1}{h_2} \frac{\partial F}{\partial u_2} \right) \right], \quad (27)$$

Formulas for general non-orthogonal surface parameters are given in (McConnell, 1931) on p. 187.

If the surface were a plane, then  $u_1 = x$  and  $u_2 = y$ , and

$$\bar{D}(U, V) = \frac{\partial U}{\partial x} \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial V}{\partial y}, \quad (28)$$

$$\Delta_2(F) = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}. \quad (29)$$



Hence we see that (24) is the two-dimensional analogue of Green's first identity in space; see, e.g., (Moritz, 1964, sec. 2.1). Therefore (24) is called Green's theorem for a surface.

If  $S$  is a closed surface, and if the curve  $C$  is contracted to a point, then  $S_1$ , taken to be the exterior of  $C$ , becomes the whole surface  $S$ , and the first integral on the right-hand side of (24) reduces to zero. As a limit there remains

$$\iint_S \bar{D}(U, V) dS = - \iint_S U \Delta_2 V dS . \quad (30)$$

Because of the symmetry of  $\bar{D}$ , also

$$\iint_S \bar{D}(U, V) dS = - \iint_S V \Delta_2 U dS . \quad (31)$$

In these identities the functions  $F, U, V$  are supposed to be continuous and twice differentiable in the region considered:  $S_1$  in (24) and  $S$  in (30) and (31).

The function

$$V = \frac{1}{l} , \quad (32)$$

where  $l$  is the distance of a variable point  $P'$  from a fixed point  $P$  on the surface, satisfies this requirement everywhere except at  $P$ , where  $1/l$  becomes infinite.

In this case it may be shown that (31) still holds:

$$\iint_S \bar{D}\left(F, \frac{1}{l}\right) dS = - \iint_S \frac{\Delta_2 F}{l} dS , \quad (33)$$

whereas (30) must be slightly modified:

$$\iint_S \bar{D}\left(F, \frac{1}{l}\right) dS = - \iint_S (F - F_P) \Delta_2 \left(\frac{1}{l}\right) dS . \quad (34)$$

As we shall see by equation (39),  $\Delta_2(1/l)$  has a strong singularity (like  $1/l^3$ ), which is, so to speak, neutralized by subtracting from the function  $U$  its value  $U_P$  at the fixed point  $P$ . To prove (34) and (35), exclude first  $P$  by a small circle  $C$  of radius  $\epsilon$ , apply (24) and let subsequently  $\epsilon \rightarrow 0$ .

These formulas hold for an arbitrary closed surface that is sufficiently smooth. Now we shall specialize these general formulas to the sphere  $r = R$ , whose line element is given by

$$ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\lambda^2),$$

so that

$$\begin{aligned} u_1 &= \theta, & u_2 &= \lambda, \\ h_1 &= R, & h_2 &= R \sin \theta. \end{aligned}$$

Then (26) and (27) become

$$\begin{aligned} \bar{D}(U, V) &= \frac{1}{R^2} \left( U_\theta V_\theta + \frac{1}{\sin^2 \theta} U_\lambda V_\lambda \right), \\ \Delta_2(F) &= \frac{1}{R^2} \left( F_\theta \cot \theta + F_{\theta\theta} + \frac{1}{\sin^2 \theta} F_{\lambda\lambda} \right), \end{aligned} \tag{35}$$

where  $U_\theta = \partial U / \partial \theta$ ,  $U_{\theta\theta} = \partial^2 U / \partial \theta^2$ , etc.

Alternatively we may refer the sphere to coordinates  $\psi$  (spherical distance from  $P$ ) and  $\alpha$  (Azimuth in  $P$ ); see Fig. 2. These coordinates  $\psi$  and  $\alpha$  exactly correspond to  $\theta$  and  $\lambda$ , the origin being now the fixed point  $P$  instead of the north pole. Hence, alternatively,

$$\begin{aligned} \bar{D}(U, V) &= \frac{1}{R^2} \left( U_\psi V_\psi + \frac{1}{\sin^2 \psi} U_\alpha V_\alpha \right), \\ \Delta_2(F) &= \frac{1}{R^2} \left( F_\psi \cot \psi + F_{\psi\psi} + \frac{1}{\sin^2 \psi} F_{\alpha\alpha} \right). \end{aligned} \tag{36}$$

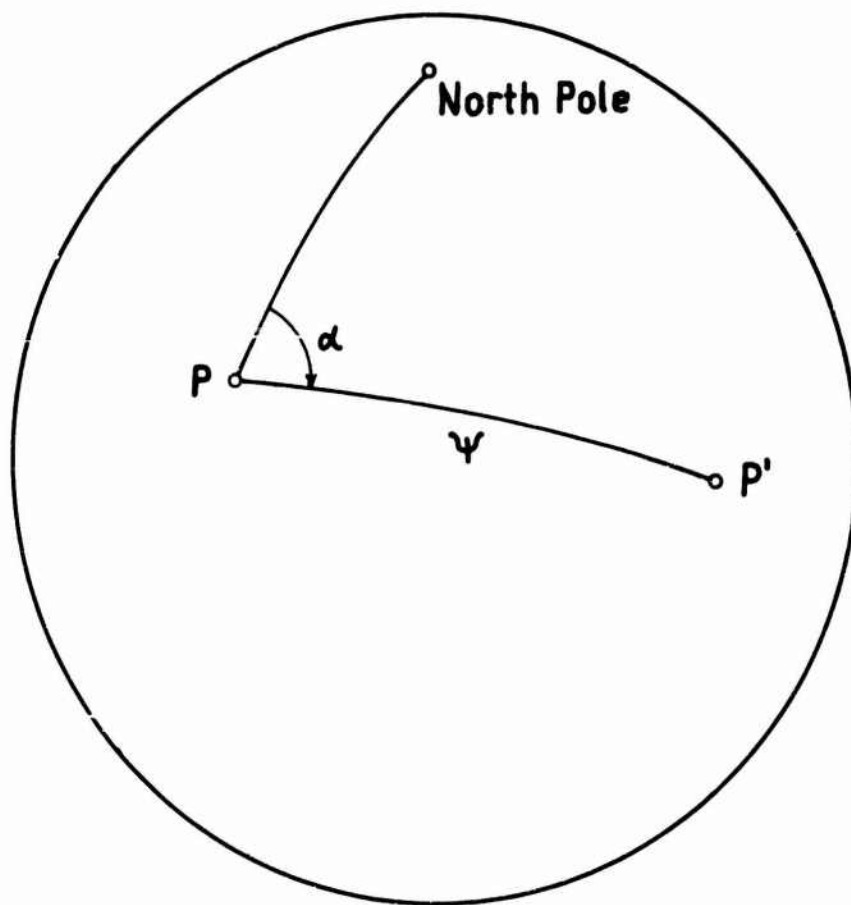


Figure 2

For the sphere we have  $l = l_0 = 2R \sin \frac{\psi}{2}$ , so that

$$\frac{1}{l_0} = \frac{1}{2R \sin \frac{\psi}{2}}, \quad (37)$$

$$\bar{D}\left(F, \frac{1}{l_0}\right) = -\frac{\sin \psi}{l_0^3} \frac{\partial F}{\partial \psi}, \quad (38)$$

$$\Delta_2\left(\frac{1}{l_0}\right) = \frac{1}{l_0^3} + \frac{1}{4R^2 l_0}. \quad (39)$$

This follows by straightforward evaluation of (36) for the particular function (37).

## 2. INTEGRAL EQUATIONS FOR THE ANOMALOUS POTENTIAL

In (Moritz, 1964) we have derived two different but equivalent integral equations, which are both due to Molodensky. We shall recall the main steps of the derivation, which we also need to derive a third alternative integral equation.

Our starting point is equation (49) of (Moritz, 1964):

$$T - \frac{1}{2\pi} \iint_S \left[ T \frac{\partial}{\partial l} \left( \frac{1}{l} \right) - \frac{1}{l} \frac{\partial T}{\partial n} \right] dS = 0. \quad (40)$$

$S$  is any known surface approximating the physical surface of the earth, such as the normal surface as defined in (Moritz, 1964, sec. 2.3) or Hirvonen's telluroid. (In the previous report cited we have used the symbol  $\Sigma$  instead of  $S$ , but this notation is somewhat awkward.) The outward directed normal to  $S$  is denoted by  $n$ , and  $l$  is the spatial distance between a fixed point  $P$  and the variable surface element  $dS$  (Fig. 3).

Consider an arbitrary function  $F$  in space. Its normal derivative may be expressed in two alternative ways; see equations (18') and (21) of (Moritz, 1964):

$$\frac{\partial F}{\partial n} = \cos \beta \frac{\partial F}{\partial h} - D(F, h) \cos \beta, \quad (41)$$

$$\frac{\partial F}{\partial n} = \frac{1}{\cos \beta} \frac{\partial F}{\partial h} - \bar{D}(F, h) \cos \beta, \quad (42)$$

where  $h$  is the elevation of the terrain and  $\beta$  is its angle of inclination.

In a local cartesian coordinate system, the  $z$ -axis pointing vertically upward, the  $x$ -axis pointing north and the  $y$ -axis pointing west, the expression  $D(F, h)$  is defined as

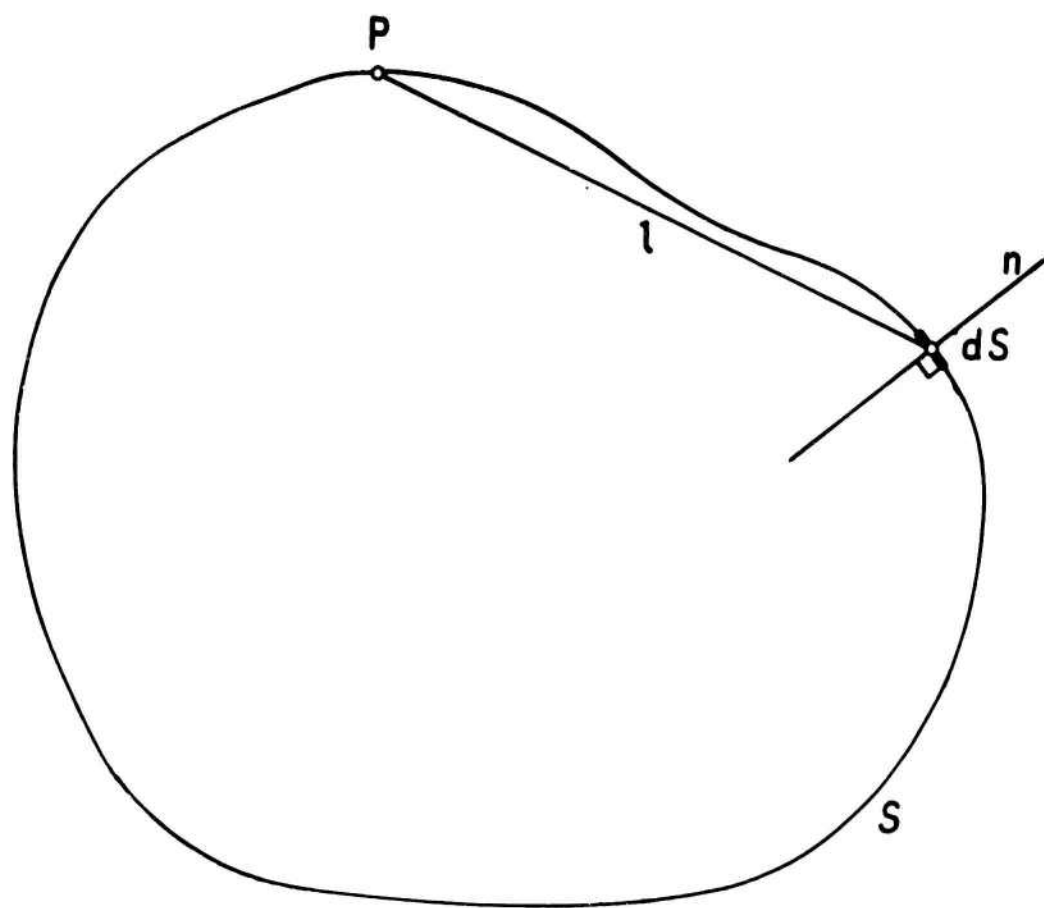


Figure 3

$$D(F, h) = \frac{\partial F}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial h}{\partial y}. \quad (43)$$

The expression  $\bar{D}(F, h)$  is defined in the same way, the derivatives along the surface

$$\frac{\partial_2 F}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial h} \frac{\partial h}{\partial x}, \quad \frac{\partial_2 F}{\partial y} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial h} \frac{\partial h}{\partial y} \quad (44)$$

replacing the horizontal derivatives  $\partial F/\partial x$  and  $\partial F/\partial y$ . Other expressions for  $D$  and  $\bar{D}$  will be given below. The vertical derivative of  $F$  is

$$\frac{\partial F}{\partial h} = \frac{\partial F}{\partial z}. \quad (45)$$

Equation (41) thus expresses the derivative of  $F$  along the surface normal  $n$  (which is not in general vertical) as a linear combination of the vertical and horizontal derivatives.

For  $F = T$  we have

$$\frac{\partial T}{\partial h} = -\Delta g + \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} T, \quad (46)$$

$$\frac{\partial T}{\partial x} = -\gamma \xi + \frac{1}{\gamma} \frac{\partial \gamma}{\partial x} T = -\gamma \xi, \quad (47a)$$

$$\frac{\partial T}{\partial y} = -\gamma \eta, \quad (47b)$$

where  $\gamma$  is normal gravity and  $\xi$  and  $\eta$  are the components of the deflection of the vertical. Hence

$$D(T, h) = \gamma (\xi \tan \beta_1 + \eta \tan \beta_2), \quad (48)$$

where

$$\tan \beta_1 = \frac{\partial h}{\partial x}, \quad \tan \beta_2 = \frac{\partial h}{\partial y} \quad (49)$$

are the inclinations of a north-south profile and an east-west profile, respectively.

Inserting (41), with  $D(T, h)$  expressed by (48), into (40) we obtain

$$\begin{aligned} T - \frac{1}{2\pi} \iint_S \left[ \frac{\partial}{\partial n} \left( \frac{1}{l} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{\cos \beta}{l} \right] T dS = \\ = \frac{1}{2\pi} \iint_S \frac{1}{l} [\Delta g - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2)] \cos \beta dS, \end{aligned} \quad (50)$$

which is our basic integral equation in the first form; it is identical with equation (48a) of (Moritz, 1964).

Secondly we use (42), again setting  $F = T$ . Molodensky (Molodenskii et al., 1962, p. 85) proved that for two functions  $U, V$

$$\iint_S \bar{D}(U, V) \cos \beta dS = - \iint_S U \Delta_2 V \cos \beta dS \quad (51a)$$

$$= - \iint_S V \Delta_2 U \cos \beta dS, \quad (51b)$$

where, in our local coordinate system, the expression  $\Delta_2 F$  is approximately defined by

$$\Delta_2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}; \quad (52)$$

more rigorous expressions will be given below. By (42) and (46) we obtain

$$\frac{1}{l} \frac{\partial T}{\partial n} = \frac{1}{l \cos \beta} \left( -\Delta g + \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} T \right) - \frac{1}{l} \bar{D}(T, h) \cos \beta. \quad (53)$$

From the definition of  $\bar{D}$  it follows that

$$\frac{1}{l} \bar{D}(T, h) = \bar{D}\left(\frac{T}{l}, h\right) - T \bar{D}\left(\frac{1}{l}, h\right). \quad (54)$$

Now we substitute (53) and (54) in (40) and apply (51a), with  $U = T/l$ . The result is



$$T - \frac{1}{2\pi} \iint_S \left[ \frac{\partial}{\partial n} \left( \frac{1}{\ell} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{1}{\ell \cos \beta} - \bar{D} \left( \frac{1}{\ell}, h \right) \cos \beta - \frac{\cos \beta}{\ell} \Delta_2 h \right] T dS = \frac{1}{2\pi} \iint_S \frac{\Delta g}{\ell \cos \beta} dS. \quad (55)$$

which is the second form of the basic integral equation; it is equation (48b) of (Moritz, 1964).

A third form is obtained by using, instead of (54), a different transformation:

$$\begin{aligned} \frac{1}{\ell} \bar{D}(T, h) &= \frac{1}{\ell} \bar{D}(T, h - h_p) \\ &= \bar{D} \left( T, \frac{h - h_p}{\ell} \right) - (h - h_p) \bar{D} \left( \frac{1}{\ell}, T \right) \\ &= \bar{D} \left( T, \frac{h - h_p}{\ell} \right) - \bar{D} \left[ (h - h_p) T, \frac{1}{\ell} \right] + T \bar{D} \left( \frac{1}{\ell}, h \right). \end{aligned} \quad (56)$$

The possibility of these manipulations follows again from the definition of  $\bar{D}$ . In agreement with (51) we have

$$\iint_S \bar{D} \left( T, \frac{h - h_p}{\ell} \right) \cos \beta dS = - \iint_S \frac{h - h_p}{\ell} \Delta_2 T \cos \beta dS, \quad (57)$$

$$\iint_S \bar{D} \left[ (h - h_p) T, \frac{1}{\ell} \right] \cos \beta dS = - \iint_S (h - h_p) T \Delta_2 \left( \frac{1}{\ell} \right) \cos \beta dS. \quad (58)$$

Again we substitute (53) in (40), but now we apply the transformation (56) and use (57) and (58). The result is

$$\begin{aligned} T - \frac{1}{2\pi} \iint_S \left[ \frac{\partial}{\partial n} \left( \frac{1}{\ell} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{1}{\ell \cos \beta} + (h - h_p) \Delta_2 \left( \frac{1}{\ell} \right) \cos \beta + \right. \\ \left. + \bar{D} \left( \frac{1}{\ell}, h \right) \cos \beta \right] T dS = \frac{1}{2\pi} \iint_S \frac{\Delta g - (h - h_p) \Delta_2 T \cos^2 \beta}{\ell \cos \beta} dS. \end{aligned} \quad (59)$$

This is a third form of the basic integral equation.

Alternative expressions may be obtained by applying (42) to the function

$1/l$ :

$$\frac{\partial}{\partial n} \left( \frac{1}{l} \right) = \frac{1}{\cos \beta} \frac{\partial}{\partial h} \left( \frac{1}{l} \right) - \bar{D} \left( \frac{1}{l}, h \right) \cos \beta. \quad (60)$$

On this substitution the basic equations (50), (55), and (59) become

$$\begin{aligned} T - \frac{1}{2\pi} \iint_S \left[ \frac{1}{\cos \beta} \frac{\partial}{\partial h} \left( \frac{1}{l} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{\cos \beta}{l} - \bar{D} \left( \frac{1}{l}, h \right) \cos \beta \right] T dS = \\ = \frac{1}{2\pi} \iint_S \frac{1}{l} [\Delta g - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2)] \cos \beta dS, \end{aligned} \quad (61)$$

$$\begin{aligned} T - \frac{1}{2\pi} \iint_S \left[ \frac{1}{\cos \beta} \frac{\partial}{\partial h} \left( \frac{1}{l} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{1}{l \cos \beta} - 2\bar{D} \left( \frac{1}{l}, h \right) \cos \beta - \right. \\ \left. - \frac{\cos \beta}{l} \Delta_2 h \right] T dS = \frac{1}{2\pi} \iint_S \frac{\Delta g}{l \cos \beta} dS, \end{aligned} \quad (62)$$

$$\begin{aligned} T - \frac{1}{2\pi} \iint_S \left[ \frac{1}{\cos \beta} \frac{\partial}{\partial h} \left( \frac{1}{l} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{1}{l \cos \beta} + (h - h_p) \Delta_2 \left( \frac{1}{l} \right) \cos \beta \right] T dS = \\ = \frac{1}{2\pi} \iint_S \frac{1}{l \cos \beta} [\Delta g - (h - h_p) \Delta_2 T \cos^2 \beta] dS. \end{aligned} \quad (63)$$

These equations are more explicit and therefore better suited for practical solution; the third form has even become simpler because the  $\bar{D}$  term cancels out in (63).

The disturbing potential  $T$  may be determined from any of these integral equations. The necessary data are:  $\Delta g$  and the deflection components  $\xi$  and  $\eta$  in (61);  $\Delta g$  only in (62); and  $\Delta g$  and  $\Delta_2 T$  (which is essentially the anomalous

gradient of gravity) in (63). Equations (61) and (62) are due to Molodensky; (63) might be new.

The solution of these equations, including practical aspects, will be discussed in sec. 4.

THE OPERATORS  $D$ ,  $\bar{D}$ , AND  $\Delta_2$ . First of all, the reader should be warned that the operators  $\bar{D}$  and  $\Delta_2$  as used in this section are not absolutely identical with those denoted by the same symbols in sec. 1, although they are closely related. Therefore, the integral formulas (30) and (51) are very similar but not identical; note the factor  $\cos \beta$  in (51).

The reason is that in the present section the differential parameters  $D$ ,  $\bar{D}$ , and  $\Delta_2$  (even when referring to a surface  $S$ ) are essentially defined in terms of a three-dimensional coordinate system. It is convenient to use some simple system of orthogonal curvilinear coordinates  $q_1, q_2, q_3$ , such that the surfaces  $q_3 = \text{const.}$  are approximately level surfaces,  $q_1$  is approximately the latitude,  $q_2$  is approximately the longitude and  $q_3$  is approximately the elevation. Then the line element will be given by

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2, \quad (64)$$

where  $h_1, h_2, h_3$  are functions of  $q_1, q_2, q_3$  (they should not be confused with the elevation  $h$ ). Then the following definitions hold (Molodenskii et al., 1962, pp. 83-85):

$$D(U, V) = \frac{h_3}{h_1^2} \frac{\partial U}{\partial q_1} \frac{\partial V}{\partial q_1} + \frac{h_3}{h_2^2} \frac{\partial U}{\partial q_2} \frac{\partial V}{\partial q_2}, \quad (65)$$

$$\bar{D}(U, V) = \frac{h_3}{h_1^2} \frac{\partial_2 U}{\partial q_1} \frac{\partial_2 V}{\partial q_1} + \frac{h_3}{h_2^2} \frac{\partial_2 U}{\partial q_2} \frac{\partial_2 V}{\partial q_2}, \quad (66)$$

$$\Delta_2(F) = \frac{1}{h_1 h_2} \left[ \frac{\partial_2}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial_2 F}{\partial q_1} \right) + \frac{\partial_2}{\partial q_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial_2 F}{\partial q_2} \right) \right]. \quad (67)$$

As we have seen,  $\partial_2 F$  denotes differentiation along the surface. If  $F$  is a function defined on the surface  $S$  only,  $F = F(q_1, q_2)$ , then

$$\frac{\partial_2 F}{\partial q_1} = \frac{\partial F(q_1, q_2)}{\partial q_1}; \quad (68a)$$

if, however,  $F$  is originally a function defined in space,  $F = F(q_1, q_2, q_3)$ , and if the surface  $S$  has the equation  $q_3 = h(q_1, q_2)$ , then

$$\begin{aligned} \frac{\partial_2 F}{\partial q_1} &= \frac{\partial F(q_1, q_2, h(q_1, q_2))}{\partial q_1} \\ &= \frac{\partial F(q_1, q_2, q_3)}{\partial q_1} + \frac{\partial F(q_1, q_2, q_3)}{\partial q_3} \frac{\partial h}{\partial q_1} \end{aligned}$$

or briefly,

$$\frac{\partial_2 F}{\partial q_1} = \frac{\partial F}{\partial q_1} + \frac{\partial F}{\partial q_3} \frac{\partial h}{\partial q_1}. \quad (68b)$$

For surface functions such as the elevation  $h$  or the reciprocal distance  $1/\ell$ , definition (68a) holds; for space functions such as the anomalous potential  $T$ , definition (68b) must be applied.

The definitions (66) and (67) reduce to (26) and (27) only when the surface  $S$  under consideration is a surface  $q_3 = \text{const.}$  (with  $h_3 \equiv 1$ , which can always be achieved by a suitable choice of coordinates  $q_3$ ). Then  $\beta = 0$  and (51) reduces to (30) or (31). Broadly speaking, this holds when  $S$  is approximately a level surface, which the physical surface of the earth obviously is not.

The conditions to be satisfied by the functions entering in (51) are the same as for (30) and (31): the functions  $U$  and  $V$  are supposed to be continuous and

twice differentiable over the whole surface  $S$ . The function  $F = 1/\ell$  has a singularity at  $P$  and does not therefore everywhere satisfy these conditions. In spite of this fact, eq. (51a) still holds for  $U = 1/\ell$ , whereas (51b) must be modified by replacing  $V$  by  $V - V_0$ , in order to neutralize the strong singularity of  $\Delta_0 (1/\ell)$ . This is in complete analogy to (33) and (34) and is proved in the same way. This fact was used in the transformations necessary to derive (55) and (59).

As an example, we consider the geodetic coordinate system  $\varphi$  (ellipsoidal geographic latitude),  $\lambda$  (ellipsoidal longitude), and  $h$  (height above the reference ellipsoid). The line element in these coordinates is given by

$$ds^2 = (M + h)^2 d\varphi^2 + (N + h)^2 \cos^2 \varphi d\lambda^2 + dh^2 \quad (69)$$

(Molodenskii et al., 1962, p. 10), where

$$M = \frac{a^2 b^2}{(a^2 \cos^2 \varphi + b^2 \sin^2 \varphi)^{3/2}} \quad (70)$$

is the meridional radius of curvature and

$$N = \frac{a^2}{(a^2 \cos^2 \varphi + b^2 \sin^2 \varphi)^{1/2}} \quad (71)$$

is the east-west radius of curvature of the ellipsoid, whose semi-axes are  $a$  and  $b$ . Hence

$$\begin{aligned} q_1 &= \varphi, & q_2 &= \lambda, & q_3 &= h; \\ h_1 &= M + h, & h_2 &= (N + h) \cos \varphi, & h_3 &= 1. \end{aligned}$$

Equation (66) thus becomes

$$\bar{D}(U, V) = \frac{1}{(M+h)^2} \frac{\partial_2 U}{\partial \varphi} \frac{\partial_2 V}{\partial \varphi} + \frac{1}{(N+h)^2 \cos^2 \varphi} \frac{\partial_2 U}{\partial \lambda} \frac{\partial_2 V}{\partial \lambda} . \quad (72)$$

For (67) we find

$$\begin{aligned} \Delta_2(F) = & \frac{1}{(M+h)(N+h)\cos\varphi} \left[ \frac{\partial_2}{\partial \varphi} \left( \frac{(N+h)\cos\varphi}{M+h} \frac{\partial_2 F}{\partial \varphi} \right) + \right. \\ & \left. + \frac{\partial_2}{\partial \lambda} \left( \frac{M+h}{(N+h)\cos\varphi} \frac{\partial_2 F}{\partial \lambda} \right) \right] . \end{aligned} \quad (73)$$

Differentiation of  $M$  and  $N$  with respect to  $\varphi$  is straightforward (their derivatives with respect to  $\lambda$  are even zero), but it should be noted that according to (68b)

$$\begin{aligned} \frac{\partial_2}{\partial \varphi} \left( \frac{(N+h)\cos\varphi}{M+h} \right) &= \frac{\partial}{\partial \varphi} \left( \frac{(N+h)\cos\varphi}{M+h} \right) + \frac{\partial}{\partial h} \left( \frac{(N+h)\cos\varphi}{M+h} \right) \frac{\partial h}{\partial \varphi} ; \\ \frac{\partial_2}{\partial \lambda} \left( \frac{M+h}{(N+h)\cos\varphi} \right) &= \frac{\partial}{\partial h} \left( \frac{M+h}{(N+h)\cos\varphi} \right) \frac{\partial h}{\partial \lambda} . \end{aligned} \quad (74)$$

If the reference ellipsoid is considered as a sphere (spherical approximation, see next section), then

$$M = N = R ,$$

and we are left with

$$\bar{D}(U, V) = \frac{1}{R^2} \left( \frac{\partial_2 U}{\partial \varphi} \frac{\partial_2 V}{\partial \varphi} + \frac{1}{\cos^2 \varphi} \frac{\partial_2 U}{\partial \lambda} \frac{\partial_2 V}{\partial \lambda} \right) , \quad (75)$$

$$\Delta_2(F) = \frac{1}{R^2} \left( - \frac{\partial_2 F}{\partial \varphi} \tan \varphi + \frac{\partial_2^2 F}{\partial \varphi^2} + \frac{1}{\cos^2 \varphi} \frac{\partial_2^2 F}{\partial \lambda^2} \right) , \quad (76)$$

in formal analogy to (35); note that for the sphere

$$\varphi = 90^\circ - \theta . \quad (77)$$

In the local cartesian coordinate system we have the expressions (43), (44)

and (52) already given.

### 3. VARIOUS APPROXIMATIONS

SPHERICAL APPROXIMATION. The geodetic integral equations become simpler and can be solved more easily by applying certain approximations which are permissible from the point of view of accuracy.

Usually, the gravity field of an ellipsoid of revolution is taken as the normal field. The flattening of any suitable reference field is small. Hence, when dealing with ellipsoidal quantities, it is convenient to use series expansions with respect to the flattening  $f$  or a similar small parameter.

Our integral equations deal with quantities of the order of  $T$ , which are very small themselves. In these integral equations it is therefore possible to neglect all terms containing  $f$ ,  $f^2$ , etc., as a factor of  $T$ . This is the spherical approximation. The error in the height anomalies  $\zeta$  (sec. 15) of this spherical approximation is thus of the order of

$$f\zeta \approx 0.003\zeta .$$

If  $\zeta = 100$  m, this amounts to 0.3 m.

A convenient visualization of the spherical approximation is furnished by plotting the heights  $h$  above the reference ellipsoid as the heights above a mean sphere of radius  $R$ ; see Fig. 4. It should be kept in mind, however, that this is only a visualization, which has no counterpart in reality. It would be plainly wrong to say that the earth is referred to a sphere instead of an ellipsoid, at least in the literal geometrical meaning.

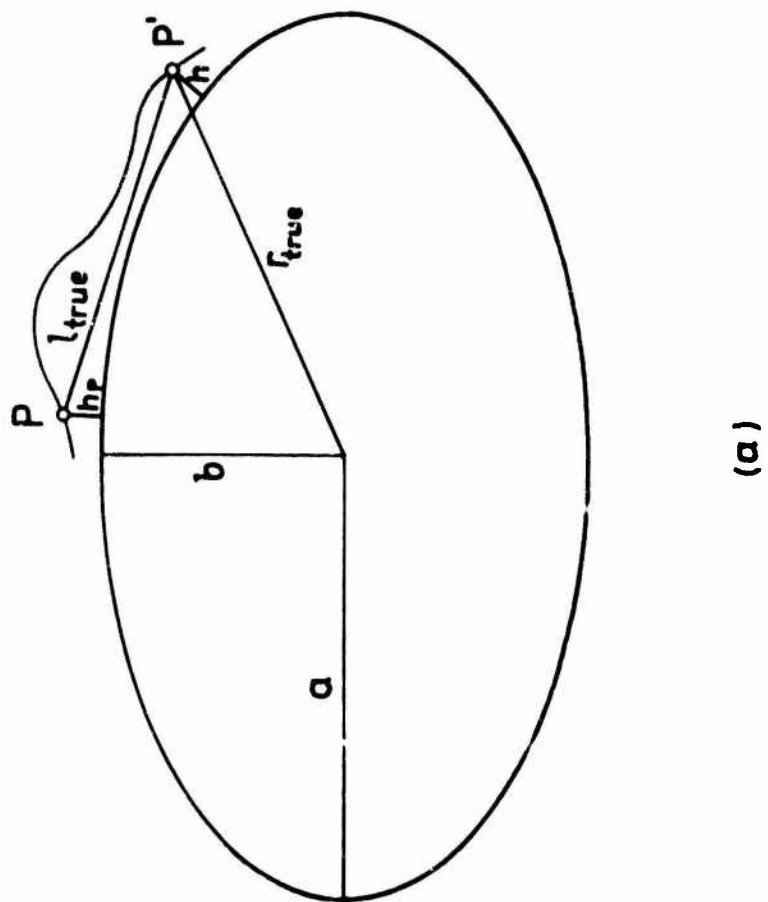
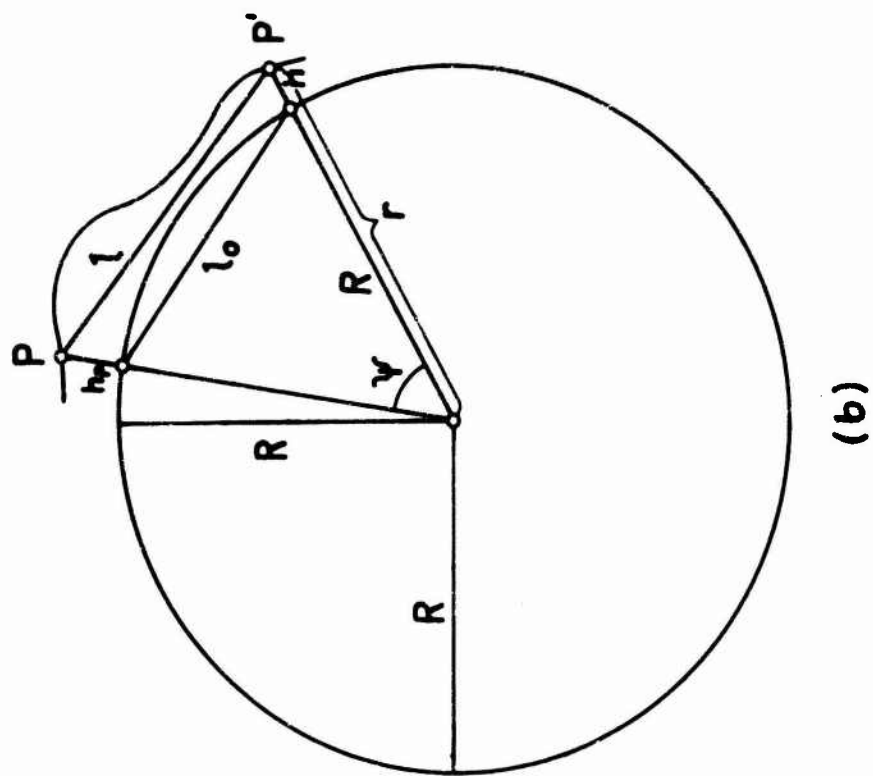


Figure 4



Consider a definite case. The true radius vector of  $P'$  (Fig. 4a) is given by

$$\begin{aligned} r_{true} &= a \left( 1 - \frac{1}{2} e'^2 \sin^2 \varphi + \dots \right) + h \\ &= R \left( 1 + \frac{1}{6} e'^2 - \frac{1}{2} e'^2 \sin^2 \varphi + \dots \right) + h, \end{aligned} \quad (78)$$

where

$$e'^2 = \frac{a^2 - b^2}{b^2}$$

is the square of the second excentricity and

$$R = \sqrt[3]{a^2 b}$$

is the mean radius of the earth. Neglecting  $e'^2$  we get

$$r = R + h \quad (78')$$

as the spherical approximation of (78). This corresponds to the geometrical interpretation of Fig. 4b.

In agreement with (78'), we have to the same approximation

$$\frac{\partial}{\partial h} = \frac{\partial}{\partial r} \quad (79)$$

and

$$\frac{1}{l} = (r^2 + r_p^2 - 2r_p r \cos \psi)^{-\frac{1}{2}} \quad (80)$$

with

$$r = R + h, \quad (81a)$$

$$r_p = R + h_p. \quad (81b)$$

Hence we obtain by differentiating (80)

$$\frac{\partial}{\partial h} \left( \frac{1}{l} \right) = \frac{\partial}{\partial r} \left( \frac{1}{l} \right) = - \frac{r - r_p \cos \psi}{l^3},$$

$$\frac{\partial}{\partial h_p} \left( \frac{1}{l} \right) = \frac{\partial}{\partial r_p} \left( \frac{1}{l} \right) = - \frac{r_p - r \cos \psi}{l^3}.$$

This is easily transformed into

$$\frac{\partial}{\partial h} \left( \frac{1}{l} \right) = - \frac{1}{2rl} + \frac{r_p^2 - r^2}{2rl^3}, \quad (82a)$$

$$\frac{\partial}{\partial h_p} \left( \frac{1}{l} \right) = - \frac{1}{2r_p l} + \frac{r^2 - r_p^2}{2r_p l^3}. \quad (82b)$$

From the spherical approximation

$$\gamma = \frac{kM}{r^2}$$

we further find

$$\frac{1}{\gamma} \frac{\partial \gamma}{\partial h} = - \frac{2}{r}. \quad (83)$$

Hence, in (62) and (63) we may set

$$\frac{\partial}{\partial h} \left( \frac{1}{l} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{1}{l} = \frac{3}{2rl} + \frac{r_p^2 - r^2}{2rl^3}. \quad (84)$$

The spherical approximation of the operators  $\bar{D}$  and  $\Delta_2$  is, of course, given by (75) and (76).

PLANAR APPROXIMATION. Comparing (78) and (78') one might object that it is illogical to neglect the term

$$\left( \frac{1}{6} - \frac{1}{2} \sin^2 \varphi \right) e'^2 R,$$

which may become as large as

$$-\frac{1}{3} e'^2 R \dot{=} -14 \text{ km} ,$$

when at the same time  $h$ , which is 8 km at its maximum, is retained.

That this objection is not valid may be seen by considering the consequences of completely neglecting  $h$ . Then the earth's surface would be represented as a sphere, and all mountains and hills would be leveled. This would amount to neglecting also the inclination of the terrain, which may attain  $45^\circ$  and more in steep hills and mountains. This is clearly inadmissible.

So, while it would be permissible to neglect the elevation as such, it is not possible to do so on account of the rapid changes of  $h$ , which cause the inclination. In other words, one may neglect  $h$  whenever it does not enter through its horizontal derivatives

$$\frac{\partial h}{\partial x} = \tan \beta_1 , \quad \frac{\partial h}{\partial y} = \tan \beta_2$$

or through similar expressions.

Such an expression is

$$\frac{h - h_p}{l_o} , \tag{85}$$

where  $l_o$  is the chord, corresponding to  $l$ , of the sphere  $r = R$  (Fig. 4b):

$$l_o = 2R \sin \frac{\psi}{2} . \tag{86}$$

For small distances  $l_o$ , the quantity  $(h - h_p)/l_o$  is of the order of the inclination  $\beta$ , as Fig. 5 shows.

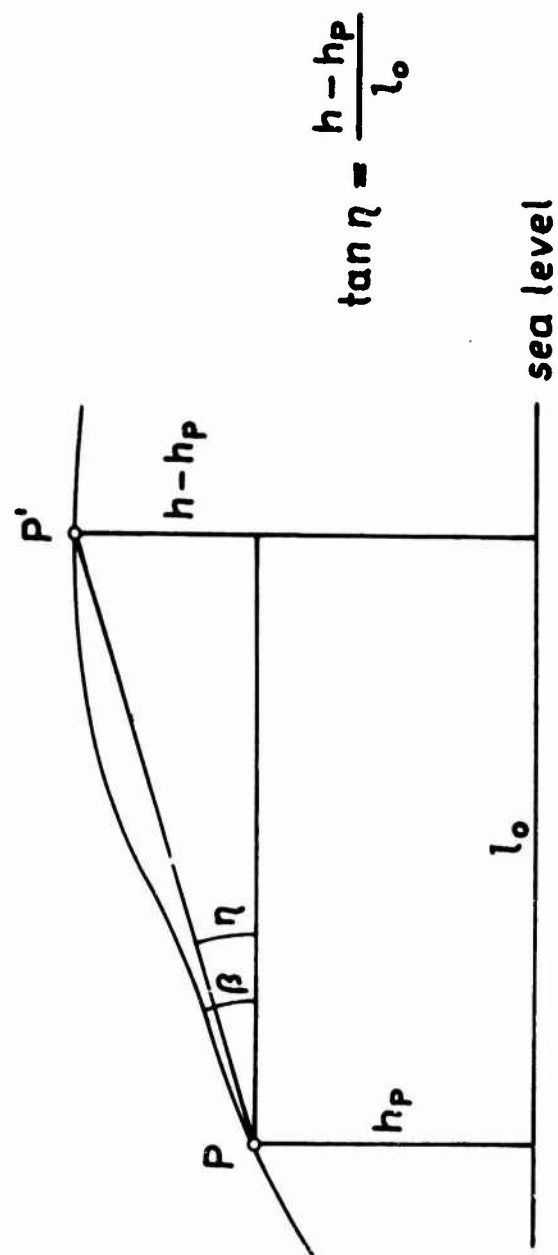


Figure 5

As a definite illustration consider (78'), which may be written

$$r = R \left( 1 + \frac{h}{R} \right). \quad (87)$$

Neglecting  $h/R$ , which at most becomes

$$\frac{h}{R} \doteq \frac{8 \text{ km}}{6 \times 10^3 \text{ km}} \doteq 0.001,$$

introduces a relative error of less than 0.1 %. However, neglecting  $\tan \beta$  when  $\beta = 45^\circ$  causes an error of 100 %!

From what has been said it appears reasonable to make an additional approximation, which we shall call planar approximation: The elevation  $h$  is neglected when it causes a relative error of only  $n/R$ . It cannot be neglected when it occurs through the expression

$$(h - h_p)/l_0$$

or indirectly through the inclination  $\beta$  (that is, through its horizontal derivatives), etc.

Let us consider some definite cases. The distance  $l$  is given by the rigorous formula

$$l^2 = r^2 + r_p^2 - 2r_p r \cos \psi. \quad (88)$$

As a spherical approximation we have  $r = R + h$ , so that

$$l^2 = (R + h)^2 + (R + h_p)^2 - 2(R + h)(R + h_p) \cos \psi.$$

This is easily transformed into

$$l^2 = 4(R + h)(R + h_p) \sin^2 \frac{\psi}{2} + (h - h_p)^2$$

and finally into

$$l^2 = l_0^2 \left[ 1 + \frac{h + h_p}{R} + \frac{h_p h}{R^2} + \left( \frac{h - h_p}{l_0} \right)^2 \right], \quad (88')$$

where  $\ell_0 = 2R \sin \frac{\psi}{2}$ . This is the spherical approximation of (88). By neglecting  $h$  except in the term

$$\left( \frac{h - h_p}{\ell_0} \right)^2$$

we obtain

$$\ell^2 = \ell_0^2 \left[ 1 + \left( \frac{h - h_p}{\ell_0} \right)^2 \right] = \ell_0^2 + (h - h_p)^2 \quad (88'')$$

as the planar approximation of (88).

As a second case, consider (84), which may also be written

$$\frac{\partial}{\partial h} \left( \frac{1}{\ell} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{1}{\ell} = \frac{3}{2r\ell} - \frac{(h - h_p)(r + r_p)}{2r\ell^3}.$$

The planar approximation is obtained by putting  $r = r_p = R$ :

$$\frac{\partial}{\partial h} \left( \frac{1}{\ell} \right) - \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \frac{1}{\ell} = \frac{3}{2R\ell} - \frac{h - h_p}{\ell^3}, \quad (89)$$

where  $\ell$  is expressed in terms of  $\ell_0$  by (88').

Neglecting  $h/R$  within the parentheses of (87) may be interpreted, in a formal mathematical manner, as letting  $R \rightarrow \infty$  (within the parentheses!), that is, as performing the formal transition from a sphere to a plane. The expression (85) will become large only if  $\ell_0$  is small, that is, in the neighborhood of  $P$ . In this neighborhood, however, the sphere may be replaced by its tangent plane. Similarly, (88'') resembles the plane formula of Pythagoras. These facts indicate that the planar approximation may be visualized in the way of Fig. 6: the elevations  $h$  above sea level or above the ellipsoid are superimposed on a plane. This interpretation furnishes a convenient name for the approximation considered.

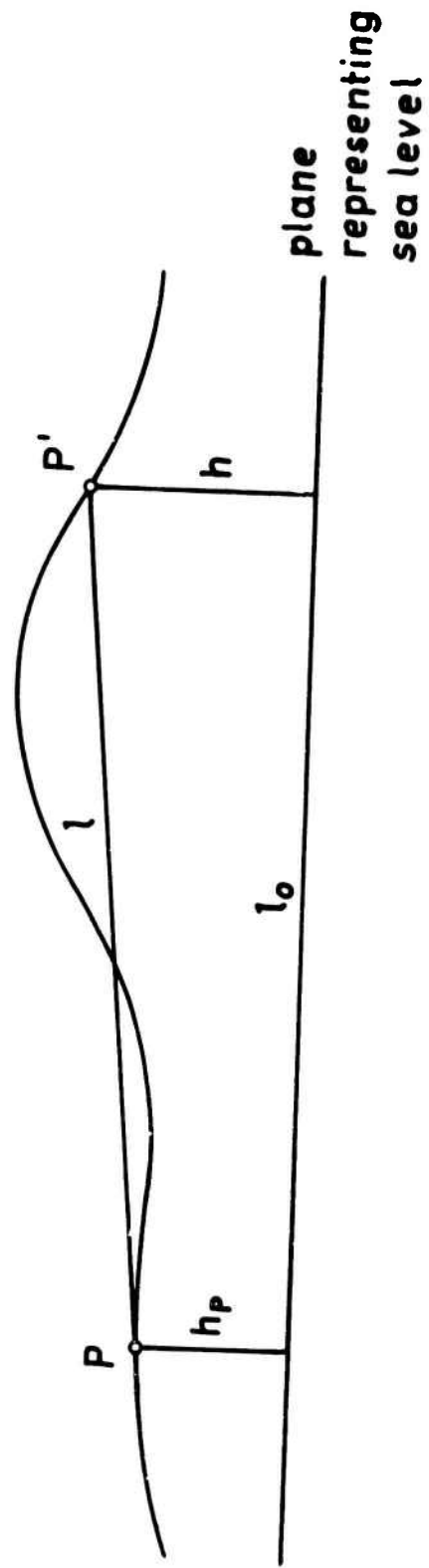


Figure 6

This name, planar approximation, however, should not be taken too literally. The transition to the plane,  $R \rightarrow \infty$ , can only be performed in expressions of the order of the elevation  $h$ , but not generally.

We may thus summarize the different approximations that are consistent from the point of view of accuracy:

Normal field (normal potential  $U$ , normal gravity  $\gamma$ , etc.): strictly ellipsoidal formulas.

Anomalous field (anomalous potential  $T$ , gravity anomalies  $\Delta g$ , etc.):

main part (not depending on  $h$ ): spherical approximation,

correction terms (order of  $h$ ): planar approximation.

In view of the conceptual difference between spherical and planar approximation it is remarkable that the accuracy requirements corresponding to the spherical approximation for  $T$ , etc., entail the planar approximation for the correction terms in a quite conclusive manner.

LINEAR APPROXIMATION. Using the planar approximation, we may expand the expressions involved into power series with respect to

$$\frac{h - h_0}{l_0}, \tan \beta, \quad (90)$$

and similar quantities. Neglecting terms of second and higher degree in these quantities and retaining only the linear terms is called linear approximation.

The linear approximation to  $\cos \beta$  is

$$\cos \beta \doteq 1, \quad (91)$$



because

$$\cos \beta = (1 + \tan^2 \beta)^{-\frac{1}{2}} = 1 - \frac{1}{2} \tan^2 \beta + \dots$$

differs from 1 only by terms of second and higher degree in  $\tan \beta$ .

As a linear approximation, we further have

$$l \doteq l_0 \quad (92)$$

for the same reason, according to (88").

Thirdly we use the surface element  $dS$ , which we shall also need subsequently, to further illustrate the meaning of the various approximations. Fig. 7, in which the profile is taken along the maximum inclination  $\beta$ , shows that, as a spherical approximation,

$$dS \cos \beta = r^2 d\sigma,$$

where  $d\sigma$  is the element of solid angle (surface element of unit sphere). Hence we have:

$$\text{spherical approximation: } dS = r^2 \sec \beta d\sigma, \quad (93a)$$

$$\text{planar approximation: } dS = R^2 \sec \beta d\sigma, \quad (93b)$$

$$\text{linear approximation: } dS = R^2 d\sigma. \quad (93c)$$

The linear approximation takes account of the main part of the effect of topography. Usually it should be sufficient in practice. Furthermore, the relation between the different solutions as considered in the present report is most obvious when the discussion is limited to the linear approximation, the relations for higher approximations becoming rapidly more complicated. Therefore, we shall here limit ourselves to linear solutions.

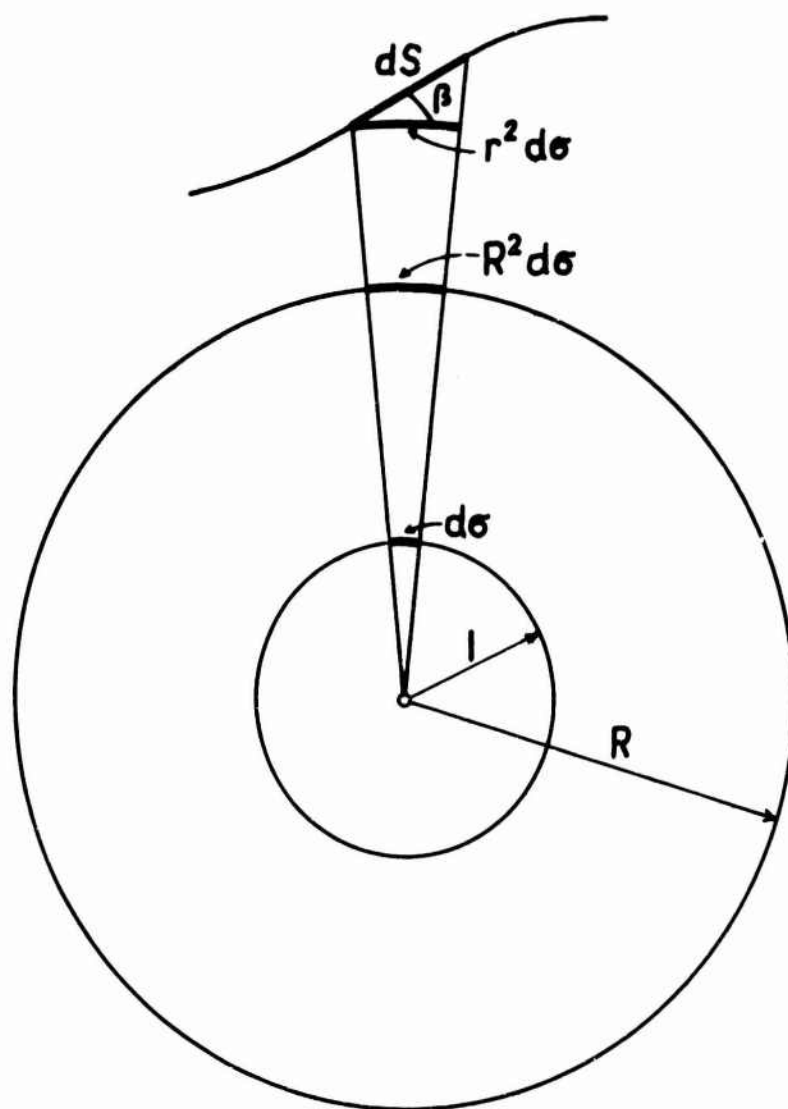


Figure 7

#### 4. LINEARIZATION AND SOLUTION OF THE INTEGRAL EQUATIONS FOR T

By means of the spherical, planar, and linear approximation, using equations (89), (91), (92), and (93), we transform the basic integral equations (61), (62), and (63) into

$$\begin{aligned} T - \frac{R^2}{2\pi} \iint_{\sigma} \left[ \frac{3}{2R\ell_0} - \frac{h - h_p}{\ell_0^3} - \bar{D}\left(\frac{1}{\ell_0}, h\right) \right] T d\sigma = \\ = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0} \left[ \Delta g - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2) \right] d\sigma, \end{aligned} \quad (94a)$$

$$\begin{aligned} T - \frac{R^2}{2\pi} \iint_{\sigma} \left[ \frac{3}{2R\ell_0} - \frac{h - h_p}{\ell_0^3} - 2\bar{D}\left(\frac{1}{\ell_0}, h\right) - \frac{\Delta_2 h}{\ell_0} \right] T d\sigma = \\ = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g}{\ell_0} d\sigma, \end{aligned} \quad (94b)$$

$$\begin{aligned} T - \frac{R^2}{2\pi} \iint_{\sigma} \left[ \frac{3}{2R\ell_0} - \frac{h - h_p}{\ell_0^3} + (h - h_p) \Delta_2 \left(\frac{1}{\ell_0}\right) \right] T d\sigma = \\ = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0} [\Delta g - (h - h_p) \Delta_2 T] d\sigma. \end{aligned} \quad (94c)$$

The operators  $\bar{D}$  and  $\Delta_2$  in these equations may be identified with those of sec. 1, defined by (26) and (27) or, as a spherical approximation, by (35). The reason is that (35) differs from (75) and (76) only by  $\partial_2 F / \partial \varphi$  being replaced by  $\partial F / \partial \varphi$ , etc. Now, for surface functions such as  $1/\ell_0$  or  $h$ , these two derivatives are identical, according to (68a). For space functions such as  $T$ , they differ by a term linear in  $h$ , according to (68b), which, when multiplied by  $h - h_p$ , becomes quadratic and therefore negligible in the linear approximations. This happens for the expression  $(h - h_p) \Delta_2 T$ ; all other operators  $\bar{D}$  and  $\Delta_2$  involve only surface

functions. This proves our assertion.

Hence we may use (38) and (39). The integrand on the right-hand side of (94c) then becomes

$$\begin{aligned} \left[ \frac{3}{2R\ell_0} - \frac{h-h_p}{\ell_0^3} + (h-h_p) \Delta_2 \left( \frac{1}{\ell_0} \right) \right] T &= \left( \frac{3}{2R\ell_0} + \frac{h-h_p}{4R^2\ell_0} \right) T = \\ &= \frac{3}{2R\ell_0} \left( 1 + \frac{1}{6} \frac{h-h_p}{R} \right) T \doteq \frac{3T}{2R\ell_0}, \end{aligned}$$

because, as a planar approximation, the term  $(h-h_p)/R$  may be neglected.

All these facts being taken into account, the integral equations (94a, b, c) become finally

$$\begin{aligned} T - \frac{3R}{4\pi} \int_{\sigma} \frac{T}{\ell_0} d\sigma &= \frac{R^2}{2\pi} \int_{\sigma} \frac{1}{\ell_0} [\Delta g - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2)] d\sigma - \\ &- \frac{R^2}{2\pi} \int_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma; \end{aligned} \quad (95a)$$

$$\begin{aligned} T - \frac{3R}{4\pi} \int_{\sigma} \frac{T}{\ell_0} d\sigma &= \frac{R^2}{2\pi} \int_{\sigma} \frac{1}{\ell_0} (\Delta g - T \Delta_2 h) d\sigma - \\ &- \frac{R^2}{2\pi} \int_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - 2 \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma; \end{aligned} \quad (95b)$$

$$T - \frac{3R}{4\pi} \int_{\sigma} \frac{T}{\ell_0} d\sigma = \frac{R^2}{2\pi} \int_{\sigma} \frac{1}{\ell_0} [\Delta g - (h-h_p) \Delta_2 T] d\sigma. \quad (95c)$$

SOLUTION. These integral equations may be written in a unified form

$$T - \frac{3R}{4\pi} \int_{\sigma} \frac{T}{\ell_0} d\sigma = \frac{R^2}{2\pi} \int_{\sigma} \frac{\Delta g + p_i}{\ell_0} d\sigma + q_i \quad (i = 1, 2, 3) \quad (96)$$

where

$$\begin{aligned}
 p_1 &= -\gamma (\xi \tan \beta_1 + \eta \tan \beta_2), \\
 q_1 &= -\frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) T \, d\sigma; \\
 p_2 &= -T \Delta_2 h, \\
 q_2 &= -\frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0^3} \left( h - h_p - 2 \sin \psi \frac{\partial h}{\partial \psi} \right) T \, d\sigma; \\
 p_3 &= -(h - h_p) \Delta_2 T, \\
 q_3 &= 0
 \end{aligned} \tag{97}$$

are small correction terms of linear order.

They are readily solved in two steps. As a first step, we disregard  $q_i$  and determine an approximation  $T_0$  of  $T$  from the equation

$$T_0 - \frac{3R}{4\pi} \iint_{\sigma} \frac{T_0}{l_0} \, d\sigma = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g + p_1}{l_0} \, d\sigma. \tag{98}$$

This is a purely spherical integral equation of type (17), whose solution according to (18) is

$$T_0 = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + p_1) S(\psi) \, d\sigma. \tag{99}$$

We then put

$$T = T_0 + T_1 \tag{100}$$

and substitute in (96), obtaining

$$T_0 - \frac{3R}{4\pi} \iint_{\sigma} \frac{T_0}{l_0} \, d\sigma + T_1 - \frac{3R}{4\pi} \iint_{\sigma} \frac{T_1}{l_0} \, d\sigma = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g + p_1}{l_0} \, d\sigma + q_1.$$

The first two terms on the left-hand side cancel with the first term on the right-hand side because of (98), and there remains

$$T_1 - \frac{3R}{4\pi} \iint_{\sigma} \frac{T_1}{L_0} d\sigma = q_1. \quad (101)$$

This integral equation for  $T_1$  is of type (21), so that its solution according to (23), replacing  $\Delta g/2\pi$  by  $q_1$ , is

$$T_1 = q_1 + \frac{3}{8\pi} \iint_{\sigma} q_1 S(\psi) d\sigma. \quad (102)$$

Combining  $T_0$  and  $T_1$  we have

$$T = \frac{R}{4\pi} \iint_{\sigma} \left( \Delta g + p_1 + \frac{3q_1}{2R} \right) S(\psi) d\sigma + q_1. \quad (103)$$

Consider now the third term between the parentheses of (103),  $3q_1/2R$ . Since  $q_1$  is divided by  $R$  in this term, it is tempting to neglect it as a planar approximation by formally letting  $R \rightarrow \infty$ . This formal argument is not entirely convincing, but the fact that  $3q_1/2R$  is negligible nevertheless holds true; the proof is rather lengthy and will be deferred to Appendix I. Hence we are left with the simple formula

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + p_1) S(\psi) d\sigma + q_1. \quad (104)$$

Using (97) we may as follows summarize the equivalent linear solutions found:

$$T = \frac{R}{4\pi} \iint_{\sigma} [\Delta g - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2)] S(\psi) d\sigma - \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{L_0^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma; \quad (105)$$

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g - T \Delta_2 h) S(\psi) d\sigma$$

$$- \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - 2 \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma ; \quad (106)$$

$$T = \frac{R}{4\pi} \iint_{\sigma} [\Delta g - (h - h_p) \Delta_2 T] S(\psi) d\sigma . \quad (107)$$

A "zero-order" approximation to  $T$  is given by Stokes' integral

$$\frac{R}{4\pi} \iint_{\sigma} \Delta g S(\psi) d\sigma ;$$

obviously this value may be used to represent  $T$  in the small correction terms

$p_1$  and  $q_1$ .

Expressions for second-order terms were given by Arnold (1959a) and Koch (1965).

## 5. THE INTEGRAL EQUATION FOR THE SURFACE LAYER

A considerably simpler integral equation was obtained by Molodensky in an indirect way, which is familiar from an application to the simpler boundary-value problems of potential theory (Kellogg, 1929).

The anomalous potential  $T$  is expressed as the potential of a surface layer, or coating, on the earth's physical surface  $S$  (or on any known surface close to it such as the telluroid):

$$T = \iint_S \frac{\varphi}{l} dS, \quad (108)$$

where  $\varphi$  is the density of the layer (it incorporates the gravitational constant).

It is known from the theory of surface layers that the potential and its tangential derivatives are continuous on the surface  $S$ , whereas the derivative along the normal,  $\partial T / \partial n$ , is discontinuous there: if we approach  $S$  from the outside, the limit of  $\partial T / \partial n$  on  $S$  is

$$\left( \frac{\partial T}{\partial n} \right)_+ = -2\pi\varphi + \iint_S \varphi \frac{\partial}{\partial n_p} \left( \frac{1}{l} \right) dS, \quad (109)$$

which is different from the value on  $S$ ,

$$\frac{\partial T}{\partial n} = \iint_S \varphi \frac{\partial}{\partial n_p} \left( \frac{1}{l} \right) dS;$$

hence the discontinuity has the value  $-2\pi\varphi$ . For another direction, which encloses the angle  $\beta$  with the normal, the discontinuity is  $-2\pi\varphi \cos \beta$ ; this is a consequence



of the discontinuity -  $2\pi\varphi$  of the normal derivative and the continuity of the tangential derivatives. Let this direction be the vertical; then  $\beta$  is the angle of inclination of the terrain. Hence the limit of the vertical derivative on  $S$  is

$$\left(\frac{\partial T}{\partial h}\right)_0 = -2\pi\varphi \cos \beta + \iint_S \varphi \frac{\partial}{\partial h_p} \left(\frac{1}{\ell}\right) dS. \quad (110)$$

By substituting (108) and (110) into the boundary condition (46) we find

$$2\pi\varphi \cos \beta - \iint_S \left[ \frac{\partial}{\partial h_p} \left(\frac{1}{\ell}\right) - \left(\frac{1}{\gamma} \frac{\partial \gamma}{\partial h}\right)_p \frac{1}{\ell} \right] \varphi dS = \Delta g, \quad (111)$$

which is to be considered as an integral equation for the unknown surface density  $\varphi$ , the gravity anomaly  $\Delta g$  at the earth's surface being given.

In agreement with our notational convention, quantities outside the integral always refer to the fixed point  $P$ ; those inside the integral refer to the variable surface element  $dS$  unless they are marked by the subscript  $P$ .

If we compare the above integral equation with those obtained previously, that is (61), (62), and (63), we see that (111) is much simpler, but we also recognize an essential difference: in (111) the differentiation  $\partial/\partial h$  is referred to the fixed point  $P$ , whereas in the former equations it was referred to the variable point  $dS$ .

SPHERICAL APPROXIMATION. In agreement with (84) we have

$$\frac{\partial}{\partial h_p} \left(\frac{1}{\ell}\right) - \left(\frac{1}{\gamma} \frac{\partial \gamma}{\partial h}\right)_p \frac{1}{\ell} = \frac{3}{2r_p \ell} + \frac{r^2 - r_p^2}{2r_p \ell^3},$$

because  $\ell$  is symmetric with respect to  $r$  and  $r_p$ . Using this expression and (93a)

we find

$$2\pi\varphi \cos \beta - \iint_{\sigma} \left( \frac{3}{2r_p l} + \frac{r^2 - r_p^2}{2r_p l^3} \right) r^2 \sec \beta \varphi d\sigma = \Delta g \quad (112)$$

as the spherical approximation to (111).

PLANAR APPROXIMATION. In agreement with (89) we have

$$\frac{\partial}{\partial h_p} \left( \frac{1}{l} \right) - \left( \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} \right)_p \frac{1}{l} = \frac{3}{2Rl} + \frac{h - h_p}{l^3},$$

and the surface element is given by (93b). Hence

$$2\pi\varphi \cos \beta - \iint_{\sigma} \left[ \frac{3R}{2l} + \frac{R^2(h - h_p)}{l^3} \right] \sec \beta \varphi d\sigma = \Delta g \quad (113)$$

is the planar approximation of our integral equation.

LINEAR APPROXIMATION. By setting  $l \doteq l_0$ ,  $\cos \beta \doteq \sec \beta \doteq 1$  we obtain

the integral equation

$$2\pi\varphi - \frac{3R}{2} \iint_{\sigma} \frac{\varphi}{l_0} d\sigma = \Delta g + R^2 \iint_{\sigma} \frac{h - h_p}{l_0^3} \varphi d\sigma \quad (114)$$

or

$$\varphi - \frac{3R}{4\pi} \iint_{\sigma} \frac{\varphi}{l_0} d\sigma = \frac{1}{2\pi} (\Delta g + G_1), \quad (115)$$

where

$$G_1 = R^2 \iint_{\sigma} \frac{h - h_p}{l_0^3} \varphi d\sigma, \quad (116)$$

and (108) becomes

$$T = R^2 \iint_{\sigma} \frac{\varphi}{l_0} d\sigma, \quad (117)$$

where, as usually,  $t_0 = 2R \sin (\psi/2)$ .

SOLUTION. Comparing (117) with (19) and (115) with (21) we see that the corresponding equations are identical if  $\Delta g$  is substituted by  $\Delta g + G_1$ . Making this substitution in (18) we therefore obtain

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G_1) S(\psi) d\sigma \quad (118)$$

as the solution of the system of equations (115) and (117).

Since  $G_1$  is a small correction term, we may in (116) approximate  $\varphi$  by (22), so that

$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \left( \Delta g + \frac{3T}{2R} \right) d\sigma. \quad (119)$$

In Appendix I it is shown that neglecting

$$\frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \left( \frac{T}{R} \right) d\sigma$$

is consistent with the planar approximation. Hence we are simply left with

$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \Delta g d\sigma. \quad (120)$$

Thus the combination of (118) and (120) constitutes another linear solution of Molodensky's problem.

A very elegant method for finding higher-order approximations, which is also applicable to other geodetic integral equations, is described in (Molodenskii et al., 1962, pp. 120-3).

## 6. BROVAR'S GENERALIZATION

Molodensky expressed the anomalous potential  $T$  as the potential of a surface layer (108):

$$T = \iint_S \frac{\varphi}{l} dS. \quad (121)$$

The formal reason why this is possible is that  $1/l$  is harmonic as a function of  $P$ , and therefore, according to the theory of linear partial differential equations,  $T$  is also harmonic (that is, a solution of Laplace's partial differential equation).

Brovar's (1963c) idea is to replace  $1/l$  by a different harmonic function  $E$ , arriving at

$$T = \iint_S \varphi E dS. \quad (122)$$

This representation is valid since  $T$  will be harmonic if  $E$  is, for the same reason as above. We may consider (122) as the potential of a generalized surface layer, and the surface function  $\varphi$  as a generalized surface density.

Then, outside the surface  $S$  we have as a spherical approximation

$$\Delta g_p = - \frac{\partial T}{\partial r_p} - \frac{2T}{r_p} = \iint_S \varphi \left( - \frac{\partial E}{\partial r_p} - \frac{2E}{r_p} \right) dS. \quad (123)$$

The function  $E$  may be selected in such a way that the "kernel"

$$K = - \frac{\partial E}{\partial r_p} - \frac{2E}{r_p} \quad (124)$$

has a suitable form.

For

$$E = \frac{1}{l} \quad (125)$$

we had, according to (112),

$$K = - \frac{3}{2r_p \ell} - \frac{r^2 - r_p^2}{2r_p \ell^3}, \quad (126)$$

that is, a linear combination of the simpler functions

$$\frac{1}{\ell} \quad \text{and} \quad \frac{r^2 - r_p^2}{\ell^3}. \quad (127)$$

We shall now try to find functions that are proportional to only one of these functions (127).

To arrive at a suitable generalization, we may start from the spherical-harmonic expansion

$$\frac{1}{\ell} = \sum_{n=0}^{\infty} \frac{r^n}{r_p^{n+1}} P_n(\cos \psi), \quad (128)$$

considered as a harmonic function of the point P in space. The series (128) remains harmonic when the individual terms are multiplied by constant coefficients  $A_n$ , provided it converges. Hence we may use the representation

$$E = \sum_{n=0}^{\infty} A_n \frac{r^n}{r_p^{n+1}} P_n(\cos \psi). \quad (129)$$

Then the kernel (124) becomes

$$K = \sum_{n=0}^{\infty} (n-1) A_n \frac{r^n}{r_p^{n+2}} P_n(\cos \psi). \quad (130)$$

According to (82b) we have

$$- \frac{r^2 - r_p^2}{r_p \ell^3} = - 2 \frac{\partial}{\partial r_p} \left( \frac{1}{\ell} \right) - \frac{1}{r_p \ell}. \quad (131)$$

By means of (128) we obtain

$$-\frac{r^2 - r_p^2}{r_p \ell^3} = \sum_{n=0}^{\infty} (2n+1) \frac{r^n}{r_p^{n+3}} P_n(\cos \psi). \quad (132)$$

Hence if we wish to have

$$K = -\frac{r^2 - r_p^2}{r_p \ell^3}, \quad (133a)$$

we must take

$$A_n = \frac{2n+1}{n-1} \quad (n \neq 1); \quad (134a)$$

for

$$K = \frac{1}{r_p \ell} \quad (133b)$$

we have

$$A_n = \frac{1}{n-1} \quad (n \neq 1). \quad (134b)$$

This is seen by comparing (130) with (132) and (128), respectively.

The coefficient  $A_1$  remains undetermined: the value  $n = 1$  is an "eigenvalue."

This gives rise to slight complications, which will be skipped over here; see

(Brovar, 1964b) and also (Moritz, 1965a).

Both cases (134a, b) will now be considered. The first case is of great practical significance, whereas the second presents several features of theoretical interest.

THE KERNEL  $(r^2 - r_p^2)/\ell^3$ . In agreement with (134a) we take

$$E_1 = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{2n+1}{n-1} \frac{r^n}{r_p^{n+3}} P_n(\cos \psi), \quad (135)$$

where the prime following the summation sign is to indicate that the term  $n = 1$  is omitted; the factor  $1/4\pi$  is irrelevant. The series (135) may be summed to give the generalized Stokes' function:

$$\begin{aligned} E_1 &= \frac{1}{4\pi} \left[ S(r_p, \psi, r) - \frac{1}{r_p} \right] \\ &= \frac{1}{4\pi} \left[ \frac{2}{\ell} - \frac{5r_p}{r_p^2} \cos \psi - \frac{3\ell}{r_p^2} - \right. \\ &\quad \left. - \frac{3r_p}{r_p^2} \cos \psi \ln \frac{r_p - r \cos \psi + i}{2r_p} \right] ; \end{aligned} \quad (136)$$

see (Moritz, 1965a, equations (50) through (54)).

The density of the generalized surface layer in this case is denoted by  $\lambda$  instead of  $\varphi$ , so that (122) takes the form

$$T = \iint_S \lambda E_1 dS . \quad (137)$$

Then the kernel will be given by (133a), multiplied by  $1/4\pi$ , so that (123) becomes

$$\frac{\partial \Gamma}{\partial r_p} + \frac{2T}{r_p} = \frac{1}{4\pi} \iint_S \lambda \frac{r^2 - r_p^2}{r_p \ell^3} dS . \quad (138)$$

According to (136), the main singularity of  $E_1$  is that of  $1/\ell$ ; the second, logarithmic, singularity has no effect in the present situation, as will be seen later. Hence the function  $E$  behaves like

$$\frac{1}{2\pi \ell} \quad (139)$$

as  $\ell \rightarrow 0$ , that is, as  $P$  approaches the surface  $S$ . Therefore, the vertical derivative

of (137) will have on  $S$  the same discontinuity as the potential of a simple layer (the function  $T$  itself will be continuous). According to (110), this discontinuity is  $-2\pi\varphi\cos\beta$ ; in agreement with (139),  $\lambda$  takes the place of  $2\pi\varphi$ , so that now the discontinuity is  $-\lambda\cos\beta$ . Hence (138) becomes on  $S$

$$\frac{\partial T}{\partial r_p} + \frac{2T}{r_p} = -\lambda\cos\beta + \frac{1}{4\pi} \iint_S \lambda \frac{r^2 - r_p^2}{r_p l^3} dS, \quad (140)$$

Since the right-hand side of this equation is equal to  $-\Delta g$ , we obtain the integral equation

$$\lambda\cos\beta - \frac{1}{4\pi} \iint_S \lambda \frac{r^2 - r_p^2}{r_p l^3} dS = \Delta g \quad (141)$$

for determining  $\lambda$  from  $\Delta g$ .

The planar approximation is

$$\lambda\cos\beta - \frac{R^2}{2\pi} \iint_{\sigma} \lambda \frac{h - h_p}{l^3} \sec\beta d\sigma = \Delta g. \quad (142)$$

Since the kernel becomes zero for the sphere ( $h = h_p = 0$ ), the second term will be small, so that this equation lends itself to an iterative solution:

$$\lambda^{(1)} = \Delta g \sec\beta, \quad (143a)$$

$$\lambda^{(1+\epsilon)} = \sec\beta \left[ \Delta g + \frac{R^2}{2\pi} \iint_{\sigma} \lambda^{(1)} \frac{h - h_p}{l^3} \sec\beta d\sigma \right]. \quad (143b)$$

We shall be satisfied here with the linear approximation

$$\lambda - \frac{R^2}{2\pi} \iint_{\sigma} \lambda \frac{h - h_p}{l^3} d\sigma = \Delta g, \quad (144)$$

of which the solution is



$$\lambda = \Delta g + \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_0}{\ell^3} \Delta g \, d\sigma. \quad (145)$$

According to (120) this is

$$\lambda = \Delta g + G_1. \quad (145')$$

Let us now consider the linear approximation to  $E_1$  if  $P$  lies on the surface  $S$  itself. To this approximation we have  $\ell = \ell_0 = 2R \sin(\psi/2)$ , so that (136) reduces to

$$E_1 = \frac{1}{4\pi} \left( \frac{2}{\ell_0} - \frac{5}{R} \cos \psi - \frac{3\ell_0}{R^2} - \frac{3}{R} \cos \psi \ln \frac{R - R \cos \psi + \ell_0}{2R} \right)$$

and finally, excluding the zero-degree harmonic, to

$$E_2 = \frac{1}{4\pi R} S(\psi) \quad (146)$$

where  $S(\psi)$  is the ordinary function of Stokes. Hence equations (137), (145'), and (146) again lead to the solution (118) considered in the preceding section.

If  $P$  lies outside the surface  $S$ , then (146) no longer holds (not even as a linear approximation), and (136) must be applied. In this case we have

$$T = \frac{R^2}{4\pi} \iint_{\sigma} (\Delta g + G_1) \left[ S(r_0, \psi, r) - \frac{1}{r_0} \right] d\sigma; \quad (147)$$

this equation provides a simple and useful formula for computing the external gravity field.

Hirvonen (1960) considered the zero-degree approximation that underlies this method: he uses a function that is essentially equivalent to (136) but disregards

the linear term  $G_1$ . Brovar (1963c, 1964b) gave the integral equation (141) and solved it by the method of Molodensky mentioned at the end of the preceding section.

THE KERNEL  $1/L$  Here we use the function

$$E_2 = \sum'_{n=0}^{\infty} \frac{1}{n-1} \frac{r^n}{r_p^{n+1}} P_n(\cos \psi), \quad (148)$$

in agreement with (134b). Incidentally, this function is related to  $E_1$  according to (135), which may be written as

$$E_1 = \frac{1}{2\pi} \sum' \frac{r^n}{r_p^{n+1}} P_n + \frac{3}{4\pi} \sum' \frac{1}{n-1} \frac{r^n}{r_p^{n+1}} P_n,$$

by

$$E_1 = \frac{1}{2\pi\ell} + \frac{3}{4\pi} E_2 - \frac{1}{2\pi} \frac{r}{r_p^2} \cos \psi. \quad (149)$$

From this fact we can easily deduce a closed expression for  $E_2$ ; using (136) we find

$$E_2 = -\frac{\ell}{r_p^2} - \frac{r}{r_p^2} \left( 1 + \ell n \frac{r_p - r \cos \psi + \ell}{2r_p} \right) \cos \psi. \quad (150)$$

The density of the generalized surface layer in this case will be denoted by  $R\mu$ , so that (122) takes the form

$$T = R \oint_S \mu E_2 dS. \quad (151)$$

Then the kernel will be given by (133b), so that (123) becomes

$$-\frac{\partial T}{\partial r_p} - \frac{2T}{r_p} = \frac{R}{r_p} \oint_S \frac{\mu}{\ell} dS. \quad (152)$$

(Strictly speaking, the kernels corresponding to  $E_1$  and  $E_2$  contain a spherical harmonic of the first degree, but this harmonic will integrate to zero in (138) and (152), so that we can disregard it here.) From the interpretation of (151) as the potential of a volume distribution it follows that neither  $T$  nor  $\partial T/\partial r_p$  will undergo a discontinuity if  $P$  crosses the surface  $S$  (p. 57). Hence even for  $P$  on  $S$  equation (152) holds, so that, with the planar approximation  $R/r_p = 1$ ,

$$\iint_S \frac{\mu}{l} dS = \Delta g \quad (153)$$

is the desired integral equation for determining  $\mu$  from  $\Delta g$ .

This integral equation is formally the simplest considered so far. This does not necessarily mean that it is the most practical. Integral equations of the first kind, such as (153), are less tractable than those of the second kind (Courant and Hilbert, 1953, p. 159). (All of our previous integral equations were of the second kind.) The linear solution, however, is easily found.

As a linear approximation we have

$$R^2 \iint_{\sigma} \frac{\mu}{l_0} d\sigma = \Delta g, \quad (154)$$

which is a strictly spherical equation. It has the form (1), so that its inversion is given by

$$2\pi\mu = \frac{\Delta g}{2R} - \frac{R^2}{2\pi} \iint_0 \frac{\Delta g - \Delta g_p}{l_0^3} d\sigma. \quad (155)$$

Comparison with (16) shows that

$$\mu = - \frac{1}{2\pi} \left( \frac{\partial \Delta g}{\partial r} + \frac{3\Delta g}{2R} \right) , \quad (156)$$

so that, apart from a constant factor and a small additional term,  $\mu$  is essentially equivalent to the anomalous vertical gradient of gravity. It is well known that this vertical gradient is much more irregular than the gravity anomaly itself. This confirms what has been said above about the tractability of (153), being an integral equation of the first kind.

It would be possible to use (155) with (151), but it is more practical to consider the linear approximation of  $E_2$  rather than its rigorous expression. We have

$$r = r_p + (h - h_p) ;$$

hence

$$E_2 = E_2^0 + \frac{\partial E_2}{\partial r} (h - h_p) ,$$

where

$$E_2^0 = \frac{1}{r_p} \sum_{n=0}^{\infty} \frac{1}{n-1} P_n (\cos \psi) \doteq \frac{1}{R} \sum_{n=1}^{\infty} \frac{1}{n-1} P_n (\cos \psi)$$

corresponds to the strictly spherical case. Differentiating (148), setting

$r = r_p = R$ , remembering the well-known spherical-harmonic expansion of

Stokes' function and neglecting very small terms we find

$$\frac{\partial E_2}{\partial r} = \frac{1}{R^2} \sum_{n=0}^{\infty} \frac{n}{n-1} P_n \doteq \frac{1}{2R^2} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n = \frac{S(\psi)}{2R^2} .$$

We substitute

$$E_2 = E_2^0 + \frac{1}{2R^2} S(\psi) (h - h_p) \quad (157)$$

in (151) and use (156), neglecting the second small term which is in agreement with the planar approximation. Thus we find

$$\begin{aligned} T &= R^3 \iint_{\sigma} \mu E_2^0 d\sigma + \frac{R}{2} \iint_{\sigma} \mu (h - h_p) S(\psi) d\sigma \\ &= R^3 \iint_{\sigma} \mu E_2^0 d\sigma - \frac{R}{4\pi} \iint_{\sigma} \frac{\partial \Delta g}{\partial h} (h - h_p) S(\psi) d\sigma. \end{aligned}$$

The first term on the right-hand side is strictly spherical and must therefore be identical with Stokes' integral:

$$R^3 \iint_{\sigma} \mu E_2^0 d\sigma = \frac{R}{4\pi} \iint_{\sigma} \Delta g S(\psi) d\sigma. \quad (158)$$

Hence we finally have

$$T = \frac{R}{4\pi} \iint_{\sigma} \left[ \Delta g - \frac{\partial \Delta g}{\partial h} (h - h_p) \right] S(\psi) d\sigma \quad (159)$$

as the linear solution of (153).

The integral equation (153), without solution, was given by Brovar (1964a).

A PHYSICAL INTERPRETATION. As Brovar (1964a) pointed out, a function similar to  $E_2$  can be interpreted as the potential of a certain volume distribution. We shall now give such an interpretation to  $E_2$  itself (Brovar's  $E_2$  is different from ours).

The potential of a volume distribution is given by

$$T = \int_{r'=0}^r \iint_{\sigma} \frac{\rho(r', \theta', \lambda')}{l} r'^n \sin \theta' d\theta' d\lambda' dr' \quad (160)$$

(the gravitational constant is taken to be unity). Assume the density  $\rho$  to have the form.

$$\rho(r', \theta', \lambda') = f(r') \nu(\theta', \lambda'). \quad (161)$$

Then (160) reduces to

$$T = \iint_{\sigma} \nu F d\sigma, \quad (162)$$

where

$$F = \int_0^r \frac{1}{l} f(r') r'^n dr'. \quad (163)$$

Using (128) this becomes

$$F = \sum_{n=0}^{\infty} \frac{P_n(\cos \psi)}{r_p^{n+1}} \int_0^r f(r') r'^{n+2} dr'. \quad (164)$$

In order to simplify the interpretation, we assume that the analogous potential  $T$  does not contain spherical harmonics of degrees zero and one. Then the sum in (164) can start with  $n = 2$ . Furthermore we set

$$f(r') = r'^{-4}. \quad (165)$$

Thus (164) becomes

$$F = \sum_{n=2}^{\infty} \frac{1}{n-1} \frac{r^{n-1}}{r_p^{n+1}} P_n(\cos \psi).$$

Comparison with (148) shows that, apart from zero-degree and first-degree

harmonics left out of consideration,

$$F = \frac{E_2}{r} \doteq \frac{E_2}{R} . \quad (166)$$

Hence (162) becomes

$$T = \frac{1}{R} \iiint_{\sigma} \nu E_2 d\sigma$$

and, with (93b)

$$T = \iiint_S \frac{\nu \cos \beta}{R^2} E_2 dS. \quad (167)$$

Comparing this with (151) we see that

$$\nu = R^4 \mu \sec \beta .$$

Hence the material density of the fictitious volume distribution that produces  $T$  is

$$\rho = \left( \frac{R}{r} \right)^4 \mu \sec \beta ; \quad (168)$$

it increases towards the center.

Hence the function  $E_2$  may be considered to correspond to the potential of a volume distribution, and the generalized Stokes' function (136) corresponds according to (149) to the linear combination of the potentials of such a volume distribution and of a surface layer. Since the potential of a volume distribution is everywhere continuous together with its first derivatives, there is no discontinuity of (152) on  $S$ , and the discontinuity of (138) is solely due to (139), which represents the surface layer.

## 7. BJERHAMMAR'S INTEGRAL EQUATION

The approach of Molodensky (sec. 5) and its generalization by Brovar (sec. 6) use surface layers on the physical surface of the earth. Bjerhammar (1964) attempts the representation of the external gravity field by a layer on the reference ellipsoid or, as a spherical approximation, on a sphere. For this purpose it is necessary that the external gravity field can be analytically continued down to sea level. This gives rise to certain theoretical and computational difficulties which, being irrelevant to the linear approximation, will not be discussed here; see (Moritz, 1964, 1966).

The external gravity field is thought to be generated by a set of fictitious gravity anomalies  $\Delta g^*$  on the sphere representing the reference ellipsoid. The actual gravity anomalies  $\Delta g$  on the earth's surface as obtained by measurement are then related to  $\Delta g^*$  by the usual "upward continuation integral" (e.g., Heiskanen and Moritz, 1967, sec. 6-8):

$$\Delta g_p = \frac{R^2(r_p^2 - R^2)}{4\pi r_p} \iint_{\sigma} \frac{\Delta g^*}{l^3} d\sigma. \quad (169)$$

The notations are evident from Fig. 8. The function  $\Delta g$  on the earth's surface being given, this equation is a linear integral equation of the first kind for  $\Delta g^*$ . The peculiar simplicity of this equation rests on the fact that the integration is now rigorously extended over a sphere. It was suggested by Bjerhammar (1964).

Once  $\Delta g^*$  has been found, all computations can be done directly, nothing but spherical formulas being involved.



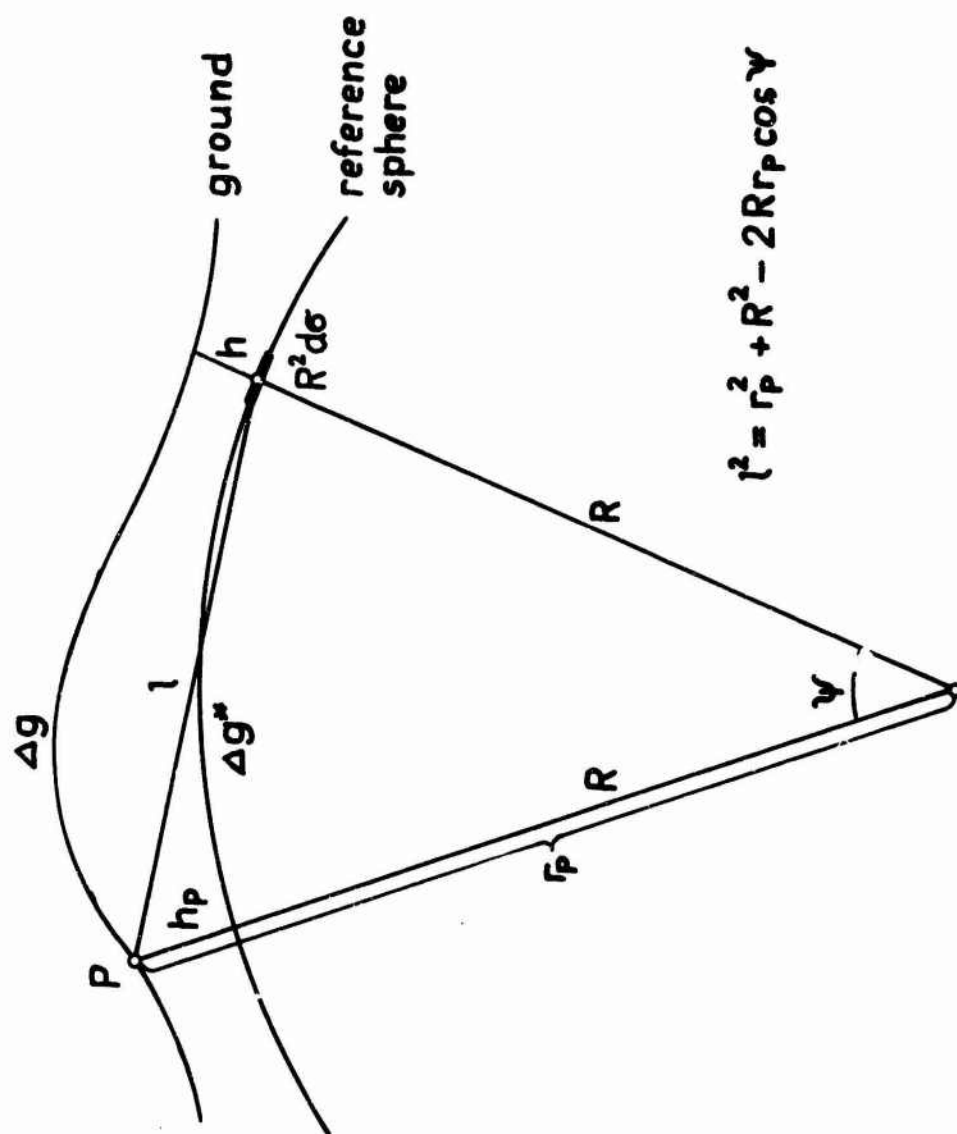


Figure 8

For solving (169) it is convenient to transform it into

$$\left(\frac{R}{r_p}\right)^2 \Delta g_p^* = \Delta g_p - \frac{R^2 (r_p^2 - R^2)}{4\pi r_p} \iint_{\sigma} \frac{\Delta g^* - \Delta g_p^*}{\ell^3} d\sigma \quad (170)$$

by a simple trick, which may be found in (Moritz, 1965b) or (Heiskanen and Moritz, 1967, sec. 8-10).

As a linear approximation we have

$$r_p^2 - R^2 = 2Rh_p, \quad \left(\frac{R}{r_p}\right)^2 = 1 - \frac{2h_p}{R};$$

hence (170) reduces to

$$\Delta g_p^* = \Delta g_p - \left( -\frac{2}{R} \Delta g_p + \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g^* - \Delta g_p^*}{\ell_o^3} d\sigma \right) h_p. \quad (171)$$

The solution of this equation, to the same accuracy, is obviously expressed by

$$\Delta g_p^* = \Delta g_p - \left( -\frac{2}{R} \Delta g_p + \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g - \Delta g_p}{\ell_o^3} d\sigma \right) h_p. \quad (172)$$

On the other hand, a linear Taylor expansion gives

$$\Delta g^* = \Delta g - \frac{\partial \Delta g}{\partial h} h. \quad (173)$$

The comparison of (173) and (172) provides an independent derivation of (16).

Arnold (1965) obtained the linear solution directly from (169) by an approach whose mathematical justification is more difficult.

To get higher approximations for  $\Delta g^*$ , the integral equation (170) may be readily solved by an iterative method similar to (143a, b). As a planar approximation, (170) reduces to

$$\Delta g^* = \Delta g - h \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g^* - \Delta g_p^*}{\ell^3} d\sigma \quad (170')$$

with the iterative solution

$$\Delta g^{*(1)} = \Delta g, \quad (174a)$$

$$\Delta g^{*(i+1)} = \Delta g - h \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g^{*(i)} - \Delta g_p^{*(i)}}{\ell^3} d\sigma. \quad (174b)$$

Details may be found in (Madkour, 1966) and (Moritz, 1966).

## B. THE LINEAR SOLUTION AS A CONSEQUENCE OF THE GRADIENT SOLUTION

### 8. THE VERTICAL GRADIENT OF GRAVITY

in Part B we shall derive the linear solutions obtained in Part A, and also others, from a simple and intuitively evident formula which uses "free-air anomalies at sea level"

$$\Delta g^* = \Delta g - \frac{\partial \Delta g}{\partial h} h .$$

As a preliminary step we consider now various expressions for the anomalous vertical gradient of gravity,  $\partial \Delta g / \partial h$ . As a spherical approximation,

$$\frac{\partial \Delta g}{\partial h} = \frac{\partial \Delta g}{\partial r} . \quad (175)$$

EXPRESSION IN TERMS OF  $\Delta g$ . In sec. 1 we have found the expression (16), which may be simplified as

$$\frac{\partial \Delta g}{\partial h} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g - \Delta g_p}{l_0^3} d\sigma , \quad (176)$$

because the small term  $(-2\Delta g/R)$ , when multiplied by  $h$  and subtracted from  $\Delta g$  according to (173), will produce a term of the order of  $h/R$  and can therefore be neglected as a planar approximation.

Since the integrand of (176) decreases rapidly with increasing distance, we may replace the sphere by its tangential plane and compute the anomalous gradient by the integral

$$\frac{\partial \Delta g}{\partial h} = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{\Delta g - \Delta g_p}{s^3} dx dy = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=0}^{\infty} \frac{\Delta g - \Delta g_p}{s^3} ds d\alpha, \quad (177)$$

which is formally extended over the whole plane; instead of  $x$  and  $y$  also polar coordinates

$$\begin{aligned} s &= \sqrt{x^2 + y^2}, \\ \alpha &= \arctan \frac{y}{x}, \end{aligned} \quad (178)$$

may be used. The origin of the coordinate system is at  $P$ .

EXPRESSION IN TERMS OF  $T$  OR  $\zeta$ . Inserting (83) into (46) and

differentiating with respect to  $r$  gives, as a spherical approximation,

$$\frac{\partial \Delta g}{\partial r} = - \frac{\partial^2 T}{\partial r^2} - \frac{2}{r} \frac{\partial T}{\partial r} + \frac{2}{r^2} T.$$

This equation is added to Laplace's equation  $\Delta T = 0$ , which in spherical coordinates  $r, \varphi, \lambda$  takes the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} - \frac{\tan \varphi}{r^2} \frac{\partial T}{\partial \varphi} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{r^2 \cos^2 \varphi} \frac{\partial^2 T}{\partial \lambda^2} = 0.$$

The result, on setting  $r = R$ , is

$$\frac{\partial \Delta g}{\partial r} = \frac{2T}{R^2} + \frac{1}{R^2} \left( -\tan \varphi \frac{\partial T}{\partial \varphi} + \frac{\partial^2 T}{\partial \varphi^2} + \frac{1}{\cos^2 \varphi} \frac{\partial^2 T}{\partial \lambda^2} \right). \quad (179)$$

The first term on the right-hand side can be neglected as a planar approximation, and according to (35), with  $\varphi = 90^\circ - \theta$ , there remains

$$\frac{\partial \Delta g}{\partial r} = \Delta_2 T. \quad (180)$$

By Bruns' theorem we have

$$T = \gamma \zeta \doteq G \zeta, \quad (181)$$

where  $G$  is a constant mean value of gravity and  $\zeta$  is the height anomaly, which for the present purpose may be identified with the geoidal undulation. Hence (180) becomes

$$\frac{\partial \Delta g}{\partial r} = G \Delta_2 \zeta. \quad (182)$$

EXPRESSION IN TERMS OF  $\xi$  AND  $\eta$ . We write (182) more explicitly as

$$\frac{\partial \Delta g}{\partial r} = \frac{G}{R^2} \left( -\tan \varphi \frac{\partial \zeta}{\partial \varphi} + \frac{\partial^2 \zeta}{\partial \varphi^2} + \frac{1}{\cos \varphi} \frac{\partial^2 \zeta}{\partial \lambda^2} \right) \quad (183)$$

and express the horizontal derivatives of  $\zeta$  in terms of the components  $\xi$  and  $\eta$  of the deflection of the vertical by the relations

$$\frac{\partial \zeta}{\partial \varphi} \doteq -R\xi, \quad \frac{\partial \zeta}{\partial \lambda} \doteq -R\eta \cos \varphi.$$

Thus we obtain

$$\frac{\partial \Delta g}{\partial r} = G \left( \frac{\xi}{R} \tan \varphi - \frac{\partial \xi}{R \partial \varphi} - \frac{\partial \eta}{R \cos \varphi \partial \lambda} \right). \quad (184)$$

It is again convenient to introduce tangential plane coordinates by

$$R d\varphi = dx, \quad R \cos \varphi d\lambda = dy, \quad (185)$$

so that

$$\frac{\partial \Delta g}{\partial r} = G \left( \frac{\xi}{R} \tan \varphi - \frac{\partial \xi}{\partial x} - \frac{\partial \eta}{\partial y} \right).$$

Since  $G\xi$  is of the order of  $\Delta g$ , the first term on the right-hand side is negligible as a planar approximation, and there remains

$$\frac{\partial \Delta g}{\partial h} = -G \left( \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right). \quad (186)$$

This formula was used by Mueller (1961).

GENERAL REFERENCE SURFACE. The vertical gradient of gravity itself is according to a theorem of Bruns given by

$$\frac{\partial g}{\partial H} = -2g\bar{J} - 2\omega^2, \quad (187)$$

$\partial/\partial H$  is the derivative along the plumb line,  $g$  is gravity,  $\bar{J}$  is the mean curvature of the level surface at the point considered, and  $\omega$  is the angular velocity of the earth's rotation.

The vertical gradient of normal gravity is correspondingly given by

$$\frac{\partial \gamma}{\partial h} = -2\gamma J - 2\omega^2, \quad (188)$$

where  $\partial/\partial h$  is the derivative along the normal plumb line, and  $J$  is the mean curvature of the normal level surface.

We divide (187) by  $g$  and (188) by  $\gamma$  and take into account that

$$gdH = \gamma dh = dW, \quad (189)$$

where  $W$  is the potential. Hence, we find

$$\frac{\partial g}{\partial W} = -2\bar{J} - 2\frac{\omega^2}{g},$$

$$\frac{\partial \gamma}{\partial W} = -2J - 2\frac{\omega^2}{\gamma},$$

so that

$$\frac{\partial \Delta g}{\partial W} = \frac{\partial g}{\partial W} - \frac{\partial \gamma}{\partial W} = -2(\bar{J} - J) - 2\omega^2\left(\frac{1}{g} - \frac{1}{\gamma}\right).$$

Using (189) we finally obtain

$$\frac{\partial \Delta g}{\partial h} = -2\gamma (\bar{J} - J) + 2\omega^2 \frac{\Delta g}{g}. \quad (190)$$

The difference of the mean curvatures of the actual and the corresponding normal level surface can be expressed in terms of  $\zeta$ , which is the vertical distance between these two surfaces. This is a purely geometrical problem, which will be solved in Appendix II. The result is

$$\bar{J} - J = -(2J^2 - K) \zeta - \frac{1}{2} \Delta_2 \zeta, \quad (191)$$

where  $K$  is the Gaussian curvature.

Equations (190) and (191) hold for an arbitrary reference surface. For a sphere, their combination reduces to (182), small terms having been neglected.

EXPRESSION IN TERMS OF HORIZONTAL DERIVATIVES OF  $\Delta g$ . We may write (176) as

$$\frac{\partial \Delta g}{\partial h} = \frac{R^2}{2\pi} \iint_{\sigma} (\Delta g - \Delta g_p) \Delta_2 \left( \frac{1}{\ell_0} \right) d\sigma, \quad (192)$$

since by (39), neglecting the second term on the right-hand side as a planar approximation,

$$\Delta_2 \left( \frac{1}{\ell_0} \right) = \frac{1}{\ell_0^3}. \quad (193)$$

The right-hand side of (192) may be transformed by Green's surface identity (34), so that we obtain

$$\frac{\partial \Delta g}{\partial h} = -\frac{R^2}{2\pi} \iint_{\sigma} \bar{D} \left( \Delta g, \frac{1}{\ell_0} \right) d\sigma. \quad (194)$$



By (38) we have

$$\bar{D} \left( \Delta g, \frac{1}{\ell_0} \right) = - \frac{\sin \psi}{\ell_0^3} \frac{\partial \Delta g}{\partial \psi} \doteq - \frac{1}{\ell_0} \frac{\partial \Delta g}{R \partial \psi} \quad (195)$$

because, as a planar approximation,  $R \sin \psi \doteq 2 R \sin \frac{\psi}{2} = \ell_0$ . We further have

$$\frac{\partial \Delta g}{R \partial \psi} = \frac{\partial \Delta g}{\partial x} \cos \alpha + \frac{\partial \Delta g}{\partial y} \sin \alpha, \quad (196)$$

where  $\partial \Delta g / \partial x$  and  $\Delta g / \partial y$  are the horizontal derivatives of  $\Delta g$  in a north-south and an east-west direction, respectively, and  $\alpha$  is the azimuth.

By means of (195) and (196), equation (194) becomes

$$\frac{\partial \Delta g}{\partial h} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left( \frac{\partial \Delta g}{\partial x} \cos \alpha + \frac{\partial \Delta g}{\partial y} \sin \alpha \right) d\sigma, \quad (197)$$

which expresses the vertical derivative of  $\Delta g$  in terms of its horizontal derivatives of first order.

Equation (194) may be further transformed by Green's identity (33).

The result is

$$\frac{\partial \Delta g}{\partial h} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta_2(\Delta g)}{\ell_0} d\sigma, \quad (198)$$

which expresses the vertical derivative of  $\Delta g$  in terms of its horizontal derivatives of first and second order.

## 9. THE GRADIENT SOLUTION

Let us consider the analytical continuation of the external gravity field down to sea level. The corresponding gravity anomaly at sea level, which will be denoted by  $\Delta g^*$ , is related to the surface free-air anomaly  $\Delta g$  by

$$\Delta g^* = \Delta g - \frac{\partial \Delta g}{\partial h} h \quad (199)$$

plus higher order terms.

This analytical continuation was called "free-air reduction to sea level" in (Moritz, 1964, sec. 6.4).

Once  $\Delta g^*$  is known, the quantities pertaining to the external gravity field can be computed by purely spherical formulas. In particular, the anomalous potential is given by Stokes' formula for external space:

$$T(r, \theta, \lambda) = \frac{R^2}{4\pi} \iint_{\sigma} \Delta g^* S(r, \psi, R) d\sigma, \quad (200)$$

where the notation of (136) is used.

This formula gives  $T$  outside the physical surface  $S$ , and also on  $S$  itself. Hence the height anomaly

$$\zeta = \frac{T}{G}$$

is found by

$$\zeta(r, \theta, \lambda) = \frac{R^2}{4\pi G} \iint_{\sigma} \Delta g^* S(r, \psi, R) d\sigma. \quad (201)$$

It is more convenient to use the linear approximation to this formula.

Since

$$r = R + h,$$

we have

$$\zeta(r, \theta, \lambda) = \zeta(R, \theta, \lambda) + \frac{\partial \zeta}{\partial h} h.$$

Now,  $\zeta(R, \theta, \lambda)$  refers to sea level and is therefore given by the ordinary Stokes' formula. Hence, taking also (199) into account, we obtain

$$\zeta = \frac{R}{4\pi G} \iint_{\sigma} \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right) S(\psi) d\sigma + \frac{\partial \zeta}{\partial h} h. \quad (202)$$

This formula gives the height anomaly  $\zeta = \zeta(r, \theta, \lambda)$  at the earth's surface.

Since various expressions for  $\partial \Delta g / \partial h$  have been given in the preceding section, there remains to determine  $\partial \zeta / \partial h$ . By differentiating the original Bruns formula  $\zeta = T/\gamma$  we find

$$\frac{\partial \zeta}{\partial h} = \frac{\partial}{\partial h} \left( \frac{T}{\gamma} \right) = \frac{1}{\gamma} \frac{\partial T}{\partial h} - \frac{1}{\gamma^2} \frac{\partial \gamma}{\partial h} T = -\frac{1}{\gamma} \left( -\frac{\partial T}{\partial h} + \frac{1}{\gamma} \frac{\partial \gamma}{\partial h} T \right),$$

so that by (46)

$$\frac{\partial \zeta}{\partial h} = -\frac{\Delta g}{\gamma} \approx -\frac{\Delta g}{G}. \quad (203)$$

Inserting this in (202) and multiplying by  $G$  we find

$$T = \frac{R}{4\pi} \iint_{\sigma} \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right) S(\psi) d\sigma - h \Delta g, \quad (204)$$

which is our basic formula, the "gradient solution."

Higher approximations to (200) may be found by the integral equation approach of sec. 7, but as we have just seen, the linear solution (204) can be derived in an elementary way.

REDUCTION TO POINT LEVEL. The sea level has no preferred position in this problem; reduction to any other level may be used. If this level has the

elevation  $h_0$  above sea level, then  $h$  must be replaced by  $h - h_0$ . If we put  $h = h_p$ , reducing to the level surface passing through the ground point  $P$  under consideration ("point level"), then  $h_0 = h_p$ , and (204) reduces to

$$T = \frac{R}{4\pi} \iint_{\sigma} \left[ \Delta g - \frac{\partial \Delta g}{\partial h} (h - h_p) \right] S(\psi) d\sigma, \quad (205)$$

since  $h$  in the last term of (204) is, according to our convention, identical with  $h_p$ , so that  $h - h_0 = h_p - h_p = 0$ .

The advantage of (205) over (204) is its simplicity, its disadvantage is that the point level varies from point to point.

## 10. ARNOLD TYPE SOLUTIONS

The starting point is (205), the vertical gradient being expressed by (190):

$$T = \frac{R}{4\pi} \iint_{\sigma} [\Delta g - (h - h_p) \Delta_a T] S(\psi) d\sigma. \quad (206)$$

We write this in the form

$$T = \frac{R}{4\pi} \iint_{\sigma} \Delta g S(\psi) d\sigma + \delta T, \quad (207)$$

where

$$\delta T = -\frac{R}{4\pi} \iint_{\sigma} (h - h_p) \Delta_a T S(\psi) d\sigma. \quad (208)$$

As a planar approximation we have

$$S(\psi) \doteq \frac{2R}{l_0} \quad (209)$$

(Appendix I) and hence

$$\delta T = -\frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0} \Delta_a T d\sigma. \quad (210)$$

By means of Green's surface identity (33) this is transformed into

$$\delta T = \frac{R^2}{2\pi} \iint_{\sigma} \bar{D} \left( \frac{h - h_p}{l_0}, T \right) d\sigma. \quad (211)$$

With

$$\begin{aligned} \bar{D} \left( \frac{h - h_p}{l_0}, T \right) &= \bar{D} \left( T, \frac{h - h_p}{l_0} \right) = \\ &= \frac{1}{l_0} \bar{D}(T, h) + (h - h_p) \bar{D} \left( T, \frac{1}{l_0} \right) \end{aligned}$$

this becomes

$$\delta T = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0} \bar{D}(T, h) d\sigma + \frac{R^2}{2\pi} \iint_{\sigma} (h - h_p) \bar{D}\left(T, \frac{1}{l_0}\right) d\sigma. \quad (212)$$

In agreement with (48) we have

$$\bar{D}(T, h) \triangleq D(T, h) = -\gamma (\xi \tan \beta_1 + \eta \tan \beta_2), \quad (213)$$

and in analogy to (195),

$$\bar{D}\left(T, \frac{1}{l_0}\right) = -\frac{\sin \psi}{l_0^3} \frac{\partial T}{\partial \psi} \triangleq -\frac{1}{l_0^3} \frac{\partial T}{R \partial \psi}, \quad (214)$$

where

$$\frac{\partial T}{R \partial \psi} \triangleq \gamma \frac{\partial \zeta}{R \partial \psi} = -\gamma (\xi \cos \alpha + \eta \sin \alpha) \quad (215)$$

because  $-\partial \zeta / R \partial \psi$  is the radial deflection component corresponding to the azimuth  $\alpha$ .

Hence (212) becomes

$$\begin{aligned} \delta T = & -\frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0} \gamma (\xi \tan \beta_1 + \eta \tan \beta_2) d\sigma \\ & + \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \gamma (\xi \cos \alpha + \eta \sin \alpha) d\sigma, \end{aligned}$$

and using (209),

$$\begin{aligned} \delta T = & -\frac{R}{4\pi} \iint_{\sigma} \gamma (\xi \tan \beta_1 + \eta \tan \beta_2) S(\psi) d\sigma \\ & + \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \gamma (\xi \cos \alpha + \eta \sin \alpha) d\sigma. \end{aligned} \quad (216)$$

Inserting this into (207) we obtain

$$T = \frac{R}{4\pi} \iint_{\sigma} [\Delta g - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2)] S(\psi) d\sigma \\ + \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{\ell_0^2} \gamma (\xi \cos \alpha + \eta \sin \alpha) d\sigma. \quad (217)$$

A slightly different form is obtained by transforming the second member of (212) using

$$(h - h_p) \bar{D} \left( T, \frac{1}{\ell_0} \right) = \bar{D} \left[ (h - h_p) T, \frac{1}{\ell_0} \right] - T \bar{D} \left( \frac{1}{\ell_0}, h \right).$$

By Green's identity (34), using (193), we find

$$\frac{R^2}{2\pi} \iint_{\sigma} \bar{D} \left[ (h - h_p) T, \frac{1}{\ell_0} \right] d\sigma = -\frac{R^2}{2\pi} \iint_{\sigma} (h - h_p) T \Delta_2 \left( \frac{1}{\ell_0} \right) d\sigma \\ = -\frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{\ell_0^3} T d\sigma. \quad (218a)$$

Corresponding to (214) we have

$$\bar{D} \left( \frac{1}{\ell_0}, h \right) = -\frac{\sin \psi}{\ell_0^3} \frac{\partial h}{\partial \psi},$$

so that

$$-\frac{R^2}{2\pi} \iint_{\sigma} T \bar{D} \left( \frac{1}{\ell_0}, h \right) d\sigma = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{\ell_0^3} \frac{\partial h}{\partial \psi} T d\sigma. \quad (218b)$$

Hence the second term in (217) may be replaced by the sum of (218a) and (218b), so that we obtain

$$T = \frac{R}{4\pi} \iint_{\sigma} [\Delta g - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2)] S(\psi) d\sigma \\ - \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma. \quad (219)$$

For later reference we note as a by-product the relation

$$\frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{\ell_0^3} (h - h_p) \frac{\partial T}{\partial \psi} d\sigma = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma, \quad (220)$$

which is the deeper reason for the equivalence of (217) and (219).

Equation (211) may be transformed in still another way. It is easily verified that the relation

$$\bar{D} \left( \frac{h - h_p}{\ell_0}, T \right) = \bar{D} \left[ (h - h_p) T, \frac{1}{\ell_0} \right] + \bar{D} \left( \frac{T}{\ell_0}, h \right) - 2 T \bar{D} \left( \frac{1}{\ell_0}, h \right)$$

holds. The second term on the right-hand side is transformed by

$$\frac{R^2}{2\pi} \iint_{\sigma} \bar{D} \left( \frac{T}{\ell_0}, h \right) d\sigma = - \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0} T \Delta_2 h d\sigma = - \frac{R}{4\pi} \iint_{\sigma} T \Delta_2 h S(\psi) d\sigma,$$

the first and third term by (218a) and (218b). Using the expression so obtained for  $\delta T$  in (207) we find

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g - T \Delta_2 h) S(\psi) d\sigma - \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - 2 \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma. \quad (221)$$

The linear solutions (206), (217), (219), and (221) are equivalent. Three of these, (206), (219), and (221), have been obtained earlier as solutions of integral equations: (107), (105), and (106). An inverse transformation, leading from the form (217) to (206), was published by Arnold (1959b). The solution (219) was derived by de Graaff-Hunter (1960).



## 11. MOLODENSKY'S SOLUTION

We start from the gradient solution (204),

$$T = \frac{R}{4\pi} \iint_{\sigma} \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right) S(\psi) d\sigma - h \Delta g, \quad (222)$$

which we try to convert into the form

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G_1) S(\psi) d\sigma; \quad (223)$$

the expression for  $G_1$  is to be determined.

Since the inverse of Stokes' formula is expressed by (11), we have from (223)

$$\Delta g + G_1 = -\frac{T}{R} - \frac{R^2}{2\pi} \iint_{\sigma} \frac{T - T_p}{\ell_0^3} d\sigma. \quad (224)$$

Inserting (222) and taking account of the fact that (11) and Stokes' integral are inverse we find

$$\Delta g + G_1 = \Delta g - \frac{\partial \Delta g}{\partial h} h + \frac{h \Delta g}{R} + \frac{R^2}{2\pi} \iint_{\sigma} \frac{h \Delta g - (h \Delta g)_p}{\ell_0^3} d\sigma.$$

The term  $\Delta g$  will cancel, and  $h \Delta g / R$  can be neglected as a linear approximation.

There remains

$$G_1 = -\frac{\partial \Delta g}{\partial h} h + \frac{R^2}{2\pi} \iint_{\sigma} \frac{h \Delta g - (h \Delta g)_p}{\ell_0^3} d\sigma. \quad (225)$$

On expressing  $\partial \Delta g / \partial h$  by (176) this becomes

$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left[ -h_p (\Delta g - \Delta g_p) + h \Delta g - h_p \Delta g_p \right] d\sigma,$$

and finally

$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \Delta g \, d\sigma. \quad (226)$$

The combination of (223) and (226) constitutes Molodensky's linear solution, which we have derived from integral equations earlier in this report (sections 5 and 6). The relation (225) between  $G_1$  and the vertical gradient was established by Molodensky et al, (1962b) in a somewhat different way.

We have presented this derivation here in order to obtain Molodensky's solution as a formal consequence of the gradient solution (204). The following derivation, however, is still shorter.

Denote by  $S'$  the level surface that passes through the point  $P$  of the earth's surface  $S$  (Fig. 9), and denote the anomalous potential on  $S'$  by  $T'$  and on  $S$  by  $T$ . Then, for two corresponding points on the same vertical such as  $B$  and  $B'$  we have

$$\begin{aligned} T &= T' + \frac{\partial T}{\partial h} (h - h_p) \\ &= T' + \left( -\Delta g - \frac{2T}{R} \right) (h - h_p) \end{aligned}$$

or, as a planar approximation,

$$T = T' - \Delta g (h - h_p).$$

This is substituted into (224) with the result

$$\Delta g + G_1 = -\frac{T'}{R} - \frac{R^2}{2\pi} \iint_{\sigma} \frac{T' - T_p}{l_0^3} \, d\sigma + \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \Delta g \, d\sigma.$$

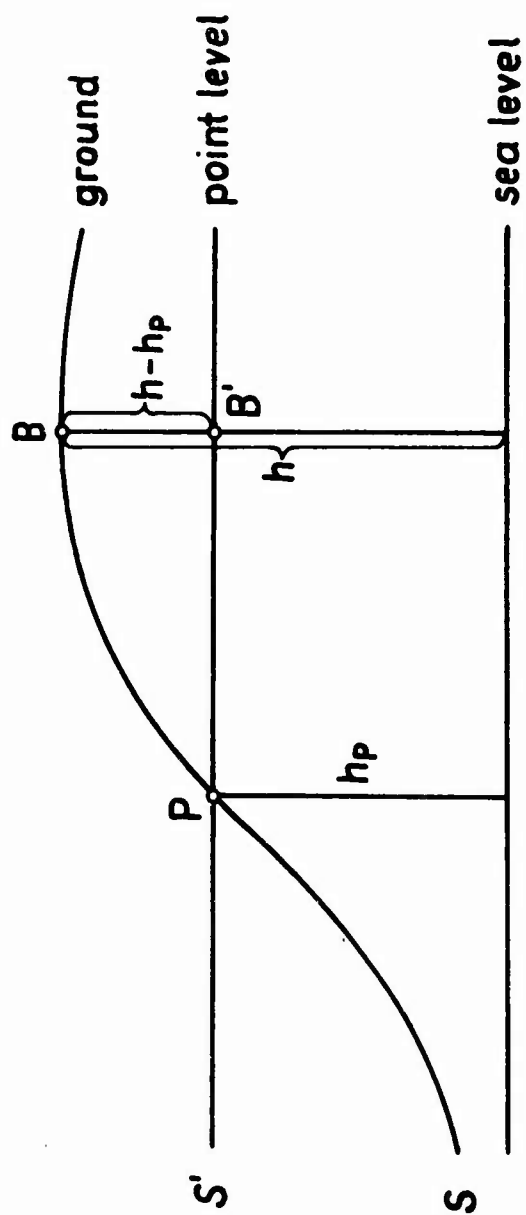


Figure 9

The first two terms on the right-hand side give the gravity anomaly  $\Delta g'$  on the level surface  $S'$ . Since the point  $P$  under consideration lies both on  $S$  and on  $S'$ , we have for this point  $\Delta g' = \Delta g$ . Hence the first two terms on the right-hand side will cancel with the first term on the left-hand side, and there remains the relation (226) which was to be derived.

## 12. TRANSFORMATIONS OF MOLODENSKY'S SOLUTION

By means of (193) we may write the Molodensky correction (226) in the form

$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} (h - h_p) \Delta g \Delta_2 \left( \frac{1}{l_0} \right) d\sigma, \quad (227)$$

which lends itself to transformation by means of Green's identity (34). The result is

$$G_1 = -\frac{R^2}{2\pi} \iint_{\sigma} \bar{D} \left[ (h - h_p) \Delta g, \frac{1}{l_0} \right] d\sigma.$$

In agreement with (195) this is

$$G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{l_0^3} \frac{\partial}{\partial \psi} \left[ (h - h_p) \Delta g \right] d\sigma. \quad (228)$$

This is an alternative form of (226). Its interest is mainly theoretical, because it serves as a starting point for further transformations.

By differentiating the product between brackets we find

$$G_1 = G_{11} + G_{12}, \quad (229)$$

where

$$G_{11} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{l_0^3} (h - h_p) \frac{\partial \Delta g}{\partial \psi} d\sigma, \quad (230)$$

$$G_{12} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{l_0^3} \Delta g \frac{\partial h}{\partial \psi} d\sigma \pm \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0^2} \Delta g \frac{\partial h}{R \partial \psi} d\sigma. \quad (231)$$

The term  $G_{12}$ , when inserted in Stokes' integral, gives rise to a term

$$T_{12} = \frac{R}{4\pi} \iint_{\sigma} G_{12} S(\psi) d\sigma \pm \frac{R^2}{2\pi} \iint_{\sigma} \frac{G_{12}}{l_0} d\sigma, \quad (232)$$

according to (209).

Being satisfied with the planar approximation, we may in these small correction terms extend the integration formally over the infinite plane.

Introducing plane polar coordinates (178) and using the notations of Fig. 10 we have

$$T_{12}(0) = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=0}^{\infty} \frac{G_{12}(s, \alpha)}{s} s \, ds \, d\alpha = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=0}^{\infty} G_{12}(s, \alpha) \, ds \, d\alpha, \quad (233)$$

$$G_{12}(s, \alpha) = \frac{1}{2\pi} \int_{\alpha'=0}^{2\pi} \int_{s'=0}^{\infty} \frac{1}{l_0^3} \frac{\partial h}{\partial l_0} \Delta g(s', \alpha') s' \, ds' \, d\alpha'. \quad (234)$$

Here

$$l_0^2 = s^2 + s'^2 - 2s s' \cos(\alpha' - \alpha); \quad (235)$$

consequently

$$\frac{\partial h}{\partial l_0} = \frac{\partial l_0}{\partial s'} \frac{\partial h}{\partial s'} + \frac{1}{s'^2} \frac{\partial l_0}{\partial \alpha'} \frac{\partial h}{\partial \alpha'}$$

becomes

$$\frac{\partial h}{\partial l_0} = \frac{s' - s \cos(\alpha' - \alpha)}{l_0} \frac{\partial h}{\partial s'} + \frac{s \sin(\alpha' - \alpha)}{l_0} \frac{\partial h}{s' \partial \alpha'}. \quad (236)$$

Substituting (234) into (233) and taking (236) into account we obtain

$$T_{12}(0) = \frac{1}{4\pi^2} \int_{\alpha=0}^{2\pi} \int_{s=0}^{\infty} \int_{\alpha'=0}^{2\pi} \int_{s'=0}^{\infty} \frac{1}{l_0^3} \left\{ [s' - s \cos(\alpha' - \alpha)] \frac{\partial h}{\partial s'} + \right. \\ \left. + s \sin(\alpha' - \alpha) \frac{\partial h}{s' \partial \alpha'} \right\} \Delta g(s', \alpha') s' \, ds' \, d\alpha' \cdot ds \, d\alpha. \quad (237)$$

We shall now perform the integration over  $s$  and  $\alpha$  first; with respect to this integration,  $\Delta g$ ,  $\partial h / \partial s'$ , and  $\partial h / s' \partial \alpha'$  are to be considered as constants. By standard methods of integration we find

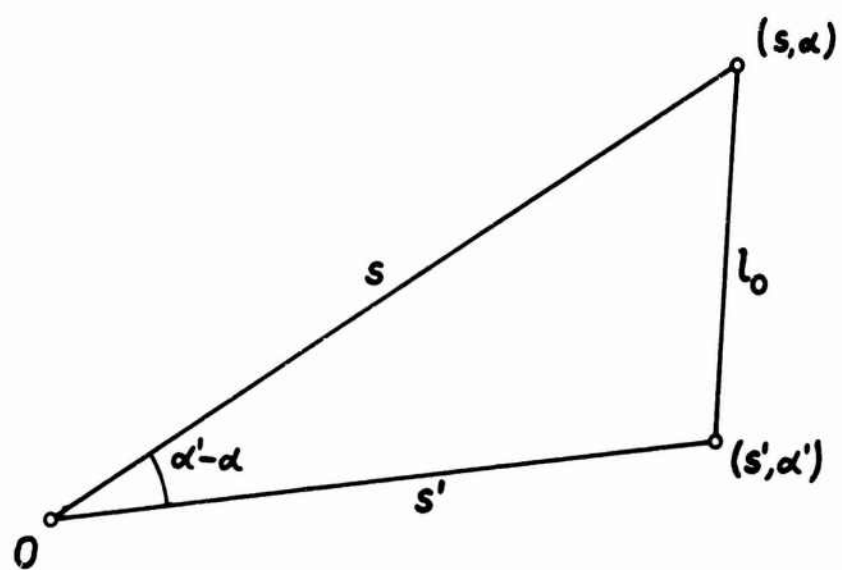


Figure 10

$$\int_{s=0}^{\infty} \frac{ds}{\ell_0^3} = \int_0^{\infty} [s^2 + s'^2 - 2s s' \cos(\alpha' - \alpha)]^{-3/2} ds = \frac{1}{2s'^2 \sin^2 \frac{\alpha' - \alpha}{2}}$$

and

$$\int_{s=0}^{\infty} \frac{s ds}{\ell_0^3} = \frac{1}{2s' \sin^2 \frac{\alpha' - \alpha}{2}},$$

so that

$$\int_0^{\infty} \frac{1}{\ell_0^3} [s' - s \cos(\alpha' - \alpha)] ds = \frac{1}{s'} \frac{1 - \cos(\alpha' - \alpha)}{2 \sin^2 \frac{\alpha' - \alpha}{2}} = \frac{1}{s'},$$

$$\int_{\alpha=0}^{2\pi} \int_{s=0}^{\infty} \frac{s' - s \cos(\alpha' - \alpha)}{\ell_0^3} ds d\alpha = \frac{2\pi}{s'}. \quad (238)$$

Similarly we find

$$\int_{\alpha=0}^{2\pi} \int_{s=0}^{\infty} \frac{s \sin(\alpha' - \alpha)}{\ell_0^3} ds d\alpha = 0 \quad (239)$$

by performing the integration with respect to  $\alpha$  first.

In view of (238) and (239), equation (237) reduces to

$$T_{12}(0) = \frac{1}{2\pi} \int_{\alpha'}^{2\pi} \int_{s'=0}^{\infty} \frac{1}{s'} \frac{\partial h}{\partial s'} \Delta g(s', \alpha') s' ds' d\alpha'.$$

If we now return to the sphere,  $s ds' d\alpha'$  becomes  $R^2 d\sigma$ , and  $s'$  becomes  $\ell_0$ ,

so that we obtain

$$T_{12} = \frac{1}{2\pi} \iint_{\sigma} \frac{1}{\ell_0} \Delta g \frac{\partial h}{R \partial \psi} d\sigma = \frac{R}{4\pi} \iint_{\sigma} \Delta g \frac{\partial h}{R \partial \psi} S(\psi) d\sigma, \quad (240)$$

according to (209).



Substituting (229) into (223) and taking (240) into account we find

$$T = \frac{R}{4\pi} \iint_{\sigma} \left( \Delta g + G_{11} + \Delta g \frac{\partial h}{R \partial \psi} \right) S(\psi) d\sigma$$

or, to the same accuracy,

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G_{11}) \left( 1 + \frac{\partial h}{R \partial \psi} \right) S(\psi) d\sigma. \quad (241)$$

The term  $\partial h / R \partial \psi$  is the radial inclination of the terrain.

An expression for  $G_{11}$  has been found in (230). It may be transformed by means of (220), in which  $T$  is replaced by  $\Delta g$ . We readily obtain

$$G_{11} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) \Delta g d\sigma. \quad (242)$$

Hence, (241) with  $G_1$  expressed by (230) or (242) constitutes another linear solution of Molodensky's problem. A form essentially equivalent to the combination of (241) and (242) was obtained by Brovar (1963a, 1964b) by solving an integral equation.

### 13. A CONNECTION WITH THE TERRAIN CORRECTION

Pellinen (1964) suggested a solution of the form

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G') S(\psi) d\sigma + t, \quad (243)$$

where

$$G' = \frac{R^2}{4\pi} \iint_{\sigma} \frac{(h - h_p)(\Delta g - \Delta g_p)}{\ell_0^3} d\sigma \quad (244)$$

and  $t$  is a correction term, which will be considered in what follows.

It will be useful to denote by the subscript  $O$  the point at which  $T$  is to be computed, and to use the subscripts 1 and 2 for distinguishing the variables of integration. Hence Molodensky's formulas (223) and (226) may be written

$$T_o = \frac{R}{4\pi} \iint_{\sigma_1} [\Delta g_1 + (G_1)_1] S(\psi_{o1}) d\sigma_1, \quad (245)$$

$$(G_1)_1 = \frac{R^2}{2\pi} \iint_{\sigma_2} \frac{h_2 - h_1}{\ell_{12}^3} \Delta g_2 d\sigma_2. \quad (246)$$

The advantage of this new notation is that now (246) can be substituted into (245) without danger of confusion:

$$T_o = \frac{R}{4\pi} \iint_{\sigma_1} \left( \Delta g_1 + \frac{R^2}{2\pi} \iint_{\sigma_2} \frac{h_2 - h_1}{\ell_{12}^3} \Delta g_2 d\sigma_2 \right) S(\psi_{o1}) d\sigma_1. \quad (247)$$

It is readily verified that the following equation, given by Pellinen (1964), is equivalent to (247):

$$\begin{aligned} T_o = & \frac{R}{4\pi} \iint_{\sigma_1} \left[ \Delta g_1 + \frac{R^2}{4\pi} \iint_{\sigma_2} \frac{(h_2 - h_1)(\Delta g_2 - \Delta g_1)}{\ell_{12}^3} d\sigma_2 \right] S(\psi_{o1}) d\sigma_1 \\ & - \frac{R^3}{16\pi^2} \iint_{\sigma_1} \iint_{\sigma_2} \Delta g_1 \frac{(h_2 - h_1)[S(\psi_{o2}) - S(\psi_{o1})]}{\ell_{12}^3} d\sigma_1 d\sigma_2. \end{aligned} \quad (248)$$

It is of the form (243). Unfortunately the expression for  $t$  -- the last integral on the right-hand side -- is impracticable since  $S(\psi_{01})$  and  $S(\psi_{02})$  are singular at the point  $O$ , and there is apparently no direct way of avoiding or neutralizing this singularity.

Therefore we shall proceed differently. We consider the gradient solution (204), the vertical gradient being expressed by (176). Using our present notation we may write it

$$T_0 = \frac{R}{4\pi} \iint_{\sigma_1} \left( \Delta g_1 - \frac{R^2}{2\pi} \iint_{\sigma_2} h_1 \frac{\Delta g_2 - \Delta g_1}{l_{12}^3} d\sigma_2 \right) S(\psi_{01}) d\sigma_1 - h_0 \Delta g_0. \quad (249)$$

We form the arithmetic mean of the equivalent equations (247) and (249) and perform some elementary manipulations. The result is

$$\begin{aligned} T_0 = & \frac{R}{4\pi} \iint_{\sigma_1} \left[ \Delta g_1 + \frac{R^2}{4\pi} \iint_{\sigma_2} \frac{(h_2 - h_1)(\Delta g_2 - \Delta g_1)}{l_{12}^3} d\sigma_2 \right. \\ & \left. + \frac{R^2}{4\pi} \left( \Delta g_1 \iint_{\sigma_2} \frac{h_2 - h_1}{l_{12}^3} d\sigma_2 - h_1 \iint_{\sigma_2} \frac{\Delta g_2 - \Delta g_1}{l_{12}^3} d\sigma_2 \right) \right] S(\psi_{01}) d\sigma_1 \\ & - \frac{1}{2} h_0 \Delta g_0. \end{aligned} \quad (250)$$

This equation has the form (243), the correction term  $t$  being given by

$$t = \frac{R}{4\pi} \iint_{\sigma} G'' S(\psi) d\sigma - \frac{1}{2} h \Delta g, \quad (251)$$

where

$$G'' = \Delta g \frac{R^2}{4\pi} \iint_{\sigma} \frac{h - h_0}{l_0^3} d\sigma - h \frac{R^2}{4\pi} \iint_{\sigma} \frac{\Delta g - \Delta g_0}{l_0^3} d\sigma; \quad (252)$$

these equations are written in our old notation.

Although the new expression for  $t$  does not contain singularities, it does not look very promising; still, under certain conditions of practical significance it reduces to a surprisingly simple form.

AN ADDITIONAL ASSUMPTION. Experience shows that the free-air anomalies  $\Delta g$  can often be represented by a linear relation

$$\Delta g = a + bh \quad (253)$$

where  $a$  and  $b$  are approximately constants. In addition,  $b$  is often approximately equal to the Bouguer gradient

$$b = 2\pi k\rho \doteq 0.11 \text{ mgal/meter} \quad (254)$$

( $k$  = gravitational constant,  $\rho$  = density), so that  $a$  is essentially nothing else than the Bouguer anomaly, which in this case is largely independent on local irregularities of topography. In statistical terms this expresses a correlation of the free-air anomaly with elevation.

By substituting (253), with constant  $a$  and  $b$ , equation (252) reduces to

$$G'' = \frac{R^2}{4\pi} a \iint_{\sigma} \frac{h - h_p}{l_o^3} d\sigma, \quad (255)$$

so that

$$\frac{2}{a} G'' = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_o^3} d\sigma.$$

To the same accuracy we have as a planar approximation

$$-\frac{2}{a} G'' = -\frac{h}{R} - \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_o^3} d\sigma.$$

By (5) and (11), the inversion of this equation is Stokes' formula:

$$h = -\frac{2}{a} \cdot \frac{R}{4\pi} \iint_{\sigma} G'' S(\psi) d\sigma,$$

so that

$$\frac{R}{4\pi} \iint_{\sigma} G'' S(\psi) d\sigma = -\frac{a}{2} h.$$

Since Stokes' integral is known to suppress the zero-degree spherical harmonic, this equation will hold only if  $h$  does not contain such a harmonic, that is if

$$\iint_{\sigma} h d\sigma = 0.$$

If this is not true, then we must obviously subtract the mean elevation

$$h_0 = \frac{1}{4\pi} \iint_{\sigma} h d\sigma$$

from  $h$ , so that we finally have

$$\frac{R}{4\pi} \iint_{\sigma} G'' S(\psi) d\sigma = -\frac{a}{2} (h - h_0). \quad (256)$$

Hence (251) becomes

$$t = -\frac{1}{2} a (h - h_0) - \frac{1}{2} h \Delta g. \quad (257)$$

Usually the zero-degree harmonic of  $\Delta g$  is assumed to vanish. By (253), this gives the condition

$$0 = \frac{1}{4\pi} \iint_{\sigma} \Delta g d\sigma = a + b h_0,$$

from which

$$a = -b h_a \quad (258)$$

and

$$\Delta g = b (h - h_a) . \quad (259)$$

Then (257) reduces to the very simple form

$$t = - \frac{1}{2} \Delta g (h - h_a) , \quad (260)$$

so that (243) becomes

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G') S(\psi) d\sigma - \frac{1}{2} \Delta g (h - h_a) . \quad (261)$$

Let us now consider the quantity  $G'$  under the assumption (253).

Substituting this relation into (244) and expressing  $b$  by (254) yields

$$G' = \frac{1}{2} k \rho R^2 \iint_{\sigma} \frac{(h - h_p)^2}{l_0^3} d\sigma , \quad (262)$$

which may be shown to be essentially identical with the conventional terrain correction for "deviation from the Bouguer plate." This interpretation, which was given by Pellinen (1964), furnishes an important link with conventional methods.

Obviously the quantity  $a$  in (253), being the Bouguer anomaly, is not a true constant for the whole earth, as was assumed in the derivation of (260). However, for (256) to hold as a planar approximation it is sufficient for  $a$  to be, loosely speaking, a local constant, or rather a quantity that varies relatively slowly. Then  $h_a$  is not the average height over the whole earth, but rather some

local average, which we may consider to be defined by (258) where  $a$  is the Bouguer anomaly and  $b$  is the Bouguer gradient.

To get an estimate of  $t$ , consider its effect on the height anomaly

$\zeta = T/G$ . This effect is by (260)

$$\delta\zeta = \frac{|t|}{G} = \frac{\Delta g}{2G} (h - h_e),$$

or by (259)

$$\delta\zeta = \frac{b}{2G} (h - h_e)^2. \quad (263)$$

If  $h - h_e = 1000$  km, which is a very extreme case, then

$$\delta\zeta = 5 \text{ cm}.$$

Hence, in those cases where  $\Delta g$  satisfies a relation (253) to a sufficient accuracy and extent for (261) to hold, we can probably always neglect the term  $t$  altogether and use the formula

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G') S(\psi) d\sigma, \quad (264)$$

that is, adding to  $\Delta g$  the terrain correction  $G'$  instead of the Molodensky correction  $G_1$ .

This solution is of particular interest from the point of view of deflections of the vertical; see sec. 15.

## C. DISCUSSION AND APPLICATIONS

### 14. HEIGHT ANOMALIES

In the preceding Parts A and B we have found expressions for the anomalous potential  $T$  at the physical surface of the earth. These formulas give directly the height anomaly  $\zeta$  as well, since by Bruns' theorem

$$\zeta = \frac{T}{\gamma} \quad (265a)$$

or, as a spherical approximation,

$$\zeta = \frac{T}{G}, \quad (265b)$$

where  $\gamma$  is normal gravity and  $G$  an average value of  $\gamma$  for the whole earth.

We shall now collect the main expressions for  $T$  previously found, it being understood that by dividing them by the constant  $G$  according to (265b) we obtain expressions for  $\zeta$ .

#### A. GRADIENT SOLUTIONS (sections 8 and 9)

$$\text{I. } T = \frac{R}{4\pi} \iint_{\sigma} \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right) S(\psi) d\sigma - h \Delta g$$

$$\text{II. } T = \frac{R}{4\pi} \iint_{\sigma} \left[ \Delta g - \frac{\partial \Delta g}{\partial h} (h - h_p) \right] S(\psi) d\sigma$$

Vertical gradient

a) by measurement

$$\text{b) } \frac{\partial \Delta g}{\partial h} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta g - \Delta g_p}{\ell_0^3} d\sigma$$



$$c) \frac{\partial \Delta g}{\partial h} = \Delta_2 T = G \Delta_2 \zeta$$

$$d) \frac{\partial \Delta g}{\partial h} = -G \left( \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right)$$

$$e) \frac{\partial \Delta g}{\partial h} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0^2} \left( \frac{\partial \Delta g}{\partial x} \cos \alpha + \frac{\partial \Delta g}{\partial y} \sin \alpha \right) d\sigma$$

$$f) \frac{\partial \Delta g}{\partial h} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta_2(\Delta g)}{l_0} d\sigma$$

#### B. MOLODENSKY TYPE SOLUTIONS (sec. 11)

$$III. \Gamma = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G_1) S(\psi) d\sigma$$

$$a) G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{l_0^3} \Delta g d\sigma$$

$$b) G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{l_0^3} \frac{\partial}{\partial \psi} [(h - h_p) \Delta g] d\sigma$$

#### C. BROVAR TYPE SOLUTIONS (sec. 12)

$$IV. T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G_{11}) \left( 1 + \frac{\partial h}{R \partial \psi} \right) S(\psi) d\sigma$$

$$a) G_{11} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{l_0^3} (h - h_p) \frac{\partial \Delta g}{\partial \psi} d\sigma$$

$$b) G_{11} = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) \Delta g d\sigma$$

#### D. ARNOLD TYPE SOLUTIONS (sec. 10)

$$V. T = \frac{R}{4\pi} \iint_{\sigma} [\Delta g - \gamma (\xi \tan \beta_1 + \eta \tan \beta_2)] S(\psi) d\sigma + K$$

$$a) K = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_p}{\ell_0^3} \gamma (\xi \cos \alpha + \eta \sin \alpha) d\sigma$$

$$b) K = - \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma$$

$$VI. T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g - T \Delta_2 h) S(\psi) d\sigma$$

$$- \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_0^3} \left( h - h_p - 2 \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma$$

#### E. PELLINEN TYPE SOLUTIONS (sec. 13)

$$VII. T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G') S(\psi) d\sigma + t$$

$$G' = \frac{R^2}{4\pi} \iint_{\sigma} \frac{(h - h_p)(\Delta g - \Delta g_p)}{\ell_0^3} d\sigma$$

$$a) t = \frac{R}{4\pi} \iint_{\sigma} G'' S(\psi) d\sigma - \frac{1}{2} h \Delta g,$$

$$G'' = \frac{R^2}{4\pi} \left( \Delta g \iint_{\sigma} \frac{h - h_p}{\ell_0^3} d\sigma - h \iint_{\sigma} \frac{\Delta g - \Delta g_p}{\ell_0^3} d\sigma \right)$$

$$b) t = - \frac{1}{2} \Delta g (h - h_p)$$

For notations the reader is referred to the sections mentioned, in which these solutions have been derived.

Without exception, these solutions consist of the original Stokes' integral and a correction, which assumes very different forms. We shall now briefly discuss these forms.

As for the gradient solutions, the basic difference between I and II is that I uses sea level (or any other fixed reference level), whereas II uses a point level that varies from point to point. Form II is somewhat simpler, but I is more suitable for large scale data processing.

The vertical gradient  $\partial\Delta g/\partial h$  entering in these formulas may be obtained from measurements (a), but the respective techniques and instruments are still at the experimental stage. The most practical form of computing it is probably from gravity anomalies using formula (b). Equation (c) expresses it in terms of the disturbing potential  $T$  or the height anomaly (or, approximately, the geoidal height)  $\zeta$ , but since second horizontal derivatives are involved, the knowledge of  $\zeta$  must be so detailed and accurate that the use of this equation is hardly feasible in practice. Equation (d), which expresses  $\partial\Delta g/\partial h$  in terms of the components  $\xi$  and  $\eta$  of the deflection of the vertical, is more useful practically; even astro-geodetic deflections may be used. Form (e) requires as data the horizontal derivatives of  $\Delta g$ , which could in principle be directly obtained by torsion balance measurements. Otherwise in (e), just as in (f), the gravity anomaly  $\Delta g$  could be used, but these two expressions are definitely less practical than (b).

Molodensky's solution III is very simple and practical when  $G_1$  is expressed in the form (a); (b) has only theoretical interest.

An advantage of the solutions of type IV over those of type III is that the integrals entering in  $G_{11}$  are better convergent than those entering in  $G_1$ .

Mathematically speaking, the integrals IV a, b are absolutely convergent, whereas the integrals III a, b only converge conditionally (Brovar, 1963a). This advantage is made up by the fact that IV is more complicated than III, the inclination of the terrain entering directly.

The Arnold type expressions have a certain theoretical significance, since they are essentially solutions of the original integral equation for the anomalous potential  $T$  (sections 2 and 4). The forms V a and VI have a certain esthetic priority since they express the correction to Stokes' integral only in terms of  $\xi$  and  $\eta$  or only in terms of  $T$ , respectively. The practical significance of these expressions is small. (We have chosen the name "Arnold type solutions" because Arnold (1959a) was among the first to find one of these solutions; later (1959b) he transformed this into the gradient solution, which he recommends for practical use.)

The interest of the solutions of Pellinen's type rests in the fact that VII contains the expression  $G'$  which (1) converges better than Molodensky's  $G_1$  and (2) has a close relationship to the conventional terrain correction. However, the general expression (a) for the correction  $t$  is impractical. Only in the case that the gravity anomalies are strongly correlated with elevation does this expression reduce to the simple form (b); in this case, furthermore, the term  $t$  may even be completely neglected in practice. Although this condition for (b) to hold will very often be fulfilled in practice, it should be kept in mind that VII b is not universally valid to a linear approximation as all the other solutions are.

As a conclusion, for practical application we are left with the gradient solutions I and II, the gradient being computed by (b) (in exceptional cases, the use of (d) might possibly be feasible) and with Molodensky's solution III a. In spite of formal similarity, the solution II is not of the form III, since  $G_1$  is a function only of its position on the earth's surface, whereas

$$- \frac{\partial \Delta g}{\partial h} (h - h_p)$$

depends, in addition, also on the computation point. For the same reason, II is somewhat inferior to I and III with respect to large-scale computations; however, this form is eminently suitable for deflections of the vertical, as we shall see in the next section. A drawback of III a is the strong and direct dependance of  $G_1$  on elevation, in particular on inclination since  $(h - h_p)/l_0$  is of the order of inclination. This drawback is avoided in the slightly longer form I, since I b does not contain the elevation as III a does.

The Brovar type solutions IV a, b show certain interesting features which make them worth to be further investigated, although they seem definitely less practical than IIIa.

The solution VII deserves further study in view of its relation to conventional methods; in the following section we shall see that it has also particular interest for computation of deflections of the vertical.

## 15. DEFLECTIONS OF THE VERTICAL

According to equation (16) of (Moritz, 1964) the components  $\xi, \eta$  of the deflection of the vertical are given by

$$\begin{aligned}\xi &= -\frac{\partial \zeta}{\partial x} - \frac{\Delta g}{\gamma} \tan \beta_1, \\ \eta &= -\frac{\partial \zeta}{\partial y} - \frac{\Delta g}{\gamma} \tan \beta_2,\end{aligned}\tag{266}$$

where, as usually,

$$\tan \beta_1 = \frac{\partial h}{\partial x}, \quad \tan \beta_2 = \frac{\partial h}{\partial y},\tag{267}$$

the derivatives  $\partial/\partial x$  and  $\partial/\partial y$  being taken along the local horizon in a northern and an eastern direction.

Each deflection component thus consists of two terms: (1) a derivative of the height anomaly and (2) a term depending on the inclination of the terrain. The derivatives  $\partial \zeta / \partial x$  and  $\partial \zeta / \partial y$  are found by differentiating any of the solutions listed in the preceding section. We shall here limit ourselves to the most important types A, B, and E; the other types may be treated in the same way.

Let us start with Molodensky's solution III:

$$\zeta = \frac{R}{4\pi G} \iint_{\sigma} (\Delta g + G_1) S(\psi) d\sigma.\tag{268}$$

Differentiation of Stokes' integral gives Vening Meinesz' formula, to which according to (266) the inclination terms must be added. The result is

$$\begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \frac{1}{4\pi G} \iint_{\sigma} (\Delta g + G_1) \frac{dS}{d\psi} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} d\sigma - \frac{\Delta g}{G} \begin{Bmatrix} \tan \beta_1 \\ \tan \beta_2 \end{Bmatrix};\tag{269}$$

in the second term,  $\gamma$  has been replaced by mean gravity  $G$ . An equivalent formula is found in (Molodenskii et al., 1962a, p. 124).

In the second place we consider the gradient solution I:

$$\zeta = \frac{R}{4\pi G} \iint_{\sigma} \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right) S(\psi) d\sigma - \frac{1}{G} h \Delta g. \quad (270)$$

Differentiation of the second term on the right-hand side of this equation and use of (267) give

$$\begin{aligned} - \frac{\partial}{\partial x} \left( - \frac{1}{G} h \Delta g \right) &= \frac{1}{G} h \frac{\partial \Delta g}{\partial x} + \frac{1}{G} \Delta g \frac{\partial h}{\partial x} \\ &= \frac{h}{G} \frac{\partial \Delta g}{\partial x} + \frac{\Delta g}{G} \tan \beta_1. \end{aligned}$$

Hence we obtain

$$\xi = \frac{1}{4\pi G} \iint_{\sigma} \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right) \frac{dS}{d\psi} \cos \alpha d\sigma + \frac{h}{G} \frac{\partial \Delta g}{\partial x} + \frac{\Delta g}{G} \tan \beta_1 - \frac{\Delta g}{G} \tan \beta_1.$$

The inclination term cancels out, and there remains

$$\left\{ \begin{matrix} \xi \\ \eta \end{matrix} \right\} = \frac{1}{4\pi G} \iint_{\sigma} \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right) \frac{dS}{d\psi} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} d\sigma + \frac{h}{G} \begin{Bmatrix} \partial \Delta g / \partial x \\ \partial \Delta g / \partial y \end{Bmatrix}. \quad (271)$$

Let us now compare (269) with (271). Theoretically, these two solutions are completely equivalent. This does not mean, however, that they are equally suited for practical application. In both cases the correction terms may assume large values. As an example, consider (269), assuming  $\Delta g = 100$  mgal and  $\beta_1 = 45^\circ$ . Then

$$\frac{\Delta g}{G} \tan \beta_1 \approx 10^{-4} \approx 20''.$$

To have a comparable situation for (271), consider an elevation  $h = 1000$  m and, again, an inclination  $\beta_1 = 45^\circ$ , assuming a linear dependence of the free-air anomaly on elevation,

$$\Delta g = a + b h ,$$

with  $b = 0.1$  mgal/m. Then

$$\frac{\partial \Delta g}{\partial x} = b \frac{\partial h}{\partial x} = b \tan \beta_1 ,$$

so that in (271)

$$\frac{h}{G} \frac{\partial \Delta g}{\partial x} \doteq 10^{-4} \doteq 20''$$

as before.

The essential difference between (269) and (271) is that the correction term in (269) is independent of elevation, whereas the correction term in (271) is proportional to elevation, so that by a suitable choice of the reference level it can be made as small as we like. By reducing the free-air anomalies to point level (sec. 9) it becomes zero; then (271) becomes simply

$$\left\{ \begin{matrix} \xi \\ \eta \end{matrix} \right\} = \frac{1}{4\pi G} \iint_{\sigma} \left[ \Delta g - \frac{\partial \Delta g}{\partial h} (h - h_p) \right] \frac{dS}{d\psi} \left\{ \begin{matrix} \cos \alpha \\ \sin \alpha \end{matrix} \right\} d\sigma . \quad (272)$$

This result could also have been obtained by formal differentiation of the gradient solution II, but the mathematical justification of this procedure is not immediately obvious.

Summarizing we may say that the correction term in both (269) and (271) show an undesirable dependence on the inclination of the terrain. In (269), this dependence is direct; in (271) it is indirect, through an approximate linear



dependence of the free-air anomalies on elevation. In (269), the effect of inclination cannot be removed or mitigated in any way; whereas in (271) it can be made smaller by reckoning the elevation from an average level approximating the terrain rather than from sea level, and it can be made zero by reckoning the elevation from point level, thus arriving at (272).

The correction to the simple Vening Meinesz' formula for  $\xi$  is according to (272) expressed by

$$- \frac{1}{4\pi G} \iint_{\sigma} \frac{\partial \Delta g}{\partial h} (h - h_p) \frac{dS}{d\psi} \cos \alpha d\sigma, \quad (273a)$$

according to (269) by

$$\frac{1}{4\pi G} \iint_{\sigma} G_1 \frac{dS}{d\psi} \cos \alpha d\sigma - \frac{\Delta g}{G} \tan \beta_1, \quad (273b)$$

and according to (271) by

$$- \frac{1}{4\pi G} \iint_{\sigma} \frac{\partial \Delta g}{\partial h} h \frac{dS}{d\psi} \cos \alpha d\sigma + \frac{h}{G} \frac{\partial \Delta g}{\partial x}. \quad (273c)$$

This correction is computed by (273a) as a single small term, whereas in (273b) and (273c) it is obtained as the difference of two larger terms computed in a different way and consequently affected by errors which may seriously endanger the result.

TRANSFORMATION OF MOLODENSKY'S FORMULA. It is, however, possible to transform Molodensky's formula (269) in such a way as to eliminate the computational problems caused by the explicit occurrence of the inclination.

Using the notation of sec. 13, we equate (273a) and (273c):

$$\begin{aligned}
 & - \frac{1}{4\pi G} \iint_{\sigma_1} \left( \frac{\partial \Delta g}{\partial h} \right)_1 (h_1 - h_0) \frac{dS}{d\psi} \cos \alpha d\sigma_1 = \\
 & = - \frac{1}{4\pi G} \iint_{\sigma_1} \left( \frac{\partial \Delta g}{\partial h} \right)_1 h_1 \frac{dS}{d\psi} \cos \alpha d\sigma + \frac{h_0}{G} \left( \frac{\partial \Delta g}{\partial x} \right)_0 ,
 \end{aligned}$$

whence

$$\frac{h_0}{G} \left( \frac{\partial \Delta g}{\partial x} \right)_0 = \frac{1}{4\pi G} \iint_{\sigma_1} h_0 \left( \frac{\partial \Delta g}{\partial h} \right)_1 \frac{dS}{d\psi} \cos \alpha d\sigma_1$$

or, by (176),

$$\frac{h_0}{G} \left( \frac{\partial \Delta g}{\partial x} \right)_0 = \frac{1}{4\pi G} \iint_{\sigma_1} \left[ \frac{R^2}{2\pi} \iint_{\sigma_2} \frac{1}{l_{12}^3} h_0 (\Delta g_2 - \Delta g_1) d\sigma_2 \right] \frac{dS}{d\psi} \cos \alpha d\sigma_1 .$$

This identity holds for arbitrary functions  $h$  and  $\Delta g$ . We may, therefore, interchange  $h$  and  $\Delta g$ ; with  $\partial h / \partial x = \tan \beta_1$  we thus obtain

$$\frac{\Delta g_0}{G} (\tan \beta_1)_0 = \frac{1}{4\pi G} \iint_{\sigma_1} \left[ \frac{R^2}{2\pi} \iint_{\sigma_2} \frac{1}{l_{12}^3} \Delta g_0 (h_2 - h_1) d\sigma_2 \right] \frac{dS}{d\psi} \cos \alpha d\sigma_1 .$$

This expression is substituted in (273b), and  $G$  is expressed by (246). The result is

$$\frac{1}{4\pi G} \iint_{\sigma_1} \left[ \frac{R^2}{2\pi} \iint_{\sigma_2} \frac{1}{l_{12}^3} (\Delta g_2 - \Delta g_0) (h_2 - h_1) d\sigma_2 \right] \frac{dS}{d\psi} \cos \alpha d\sigma ,$$

which is a correction term to be added to the simple Vening Meinesz formula for  $\xi$ ; it is equivalent to any of the expressions (273).

Hence we obtain, in our usual notation,

$$\left\{ \begin{array}{c} \xi \\ \eta \end{array} \right\} = \frac{1}{4\pi G} \iint_{\sigma} (\Delta g + \bar{G}_1) \frac{dS}{d\psi} \left\{ \begin{array}{c} \cos \alpha \\ \sin \alpha \end{array} \right\} d\sigma , \quad (274)$$

where

$$\bar{G}_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{h - h_0}{l_0^3} (\Delta g - \Delta g_0) d\sigma \quad (275)$$

is a new correction term which differs from the Molodensky correction  $G_1$  by  $\Delta g - \Delta g_A$  taking the place of  $\Delta g$ . Here  $\Delta g_A$  is the gravity anomaly at the point at which  $\xi$  and  $\eta$  are to be computed, whereas  $\Delta g_p$  would be the anomaly at the point to which  $G_1$  refers.

USE OF TERRAIN CORRECTION. We shall finally consider the solution VII b of the preceding section:

$$\zeta = \frac{R}{4\pi G} \iint_{\sigma} (\Delta g + G') S(\psi) d\sigma - \frac{1}{2G} \Delta g (h - h_A), \quad (276)$$

which holds when there is a sufficiently strong correlation of the free-air anomalies with elevation; then

$$G' = \frac{R^2}{4\pi} \iint_{\sigma} \frac{(h - h_p) (\Delta g - \Delta g_p)}{\ell_0^3} d\sigma$$

will be approximately equal to the terrain correction (sec. 13).

Differentiating the second term on the right-hand side of (276) gives

$$\begin{aligned} - \frac{\partial}{\partial x} \left[ - \frac{1}{2G} \Delta g (h - h_A) \right] &= \\ &= \frac{1}{2G} \frac{\partial \Delta g}{\partial x} (h - h_A) + \frac{1}{2G} \Delta g \frac{\partial h}{\partial x}. \end{aligned}$$

By (259) this becomes

$$\begin{aligned} &= \frac{b}{2G} \frac{\partial h}{\partial x} (h - h_A) + \frac{b}{2G} (h - h_A) \frac{\partial h}{\partial x} \\ &= \frac{1}{G} b (h - h_A) \frac{\partial h}{\partial x} = \frac{\Delta g}{G} \frac{\partial h}{\partial x} = \frac{\Delta g}{G} \tan \beta_1. \end{aligned}$$

This term cancels precisely with the second term on the right-hand side of the first equation of (266), so that there remains

$$\begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \frac{1}{4\pi G} \iint_{\sigma} (\Delta g + G) \frac{dS}{d\psi} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} d\sigma. \quad (277)$$

REVIEW. We shall now collect the results so far obtained and classify them in the same way as the solutions of sec. 14, but prefixing the letter D to the roman numerals, so that the solution DI for deflections corresponds to the solution I for T or  $\zeta$ .

#### A. GRADIENT SOLUTIONS

$$\text{DI.} \quad \xi = \frac{1}{4\pi G} \iint_{\sigma} \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right) \frac{dS}{d\psi} \cos \alpha d\sigma + \frac{h}{G} \frac{\partial \Delta g}{\partial x}$$

$$\text{DII.} \quad \xi = \frac{1}{4\pi G} \iint_{\sigma} \left[ \Delta g - \frac{\partial \Delta g}{\partial h} (h - h_A) \right] \frac{dS}{d\psi} \cos \alpha d\sigma$$

#### B. MOLODENSKY TYPE SOLUTIONS

$$\text{DIII.} \quad \xi = \frac{1}{4\pi G} \iint_{\sigma} (\Delta g + G_1) \frac{dS}{d\psi} \cos \alpha d\sigma - \frac{\Delta g}{G} \tan \beta_1$$

$$\text{where } G_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{(h - h_p) \Delta g}{l_o^3} d\sigma$$

$$\text{DIII'.} \quad \xi = \frac{1}{4\pi G} \iint_{\sigma} (\Delta g + \bar{G}_1) \frac{dS}{d\psi} \cos \alpha d\sigma$$

$$\text{where } \bar{G}_1 = \frac{R^2}{2\pi} \iint_{\sigma} \frac{(h - h_p) (\Delta g - \Delta g_A)}{l_o^3} d\sigma$$

#### E. PELLINEN TYPE SOLUTION

$$\text{DVII b.} \quad \xi = \frac{1}{4\pi G} \iint_{\sigma} (\Delta g + G') \frac{dS}{d\psi} \cos \alpha d\sigma$$

$$\text{where } G' = \frac{R^2}{4\pi} \iint_{\sigma} \frac{(h - h_p) (\Delta g - \Delta g_p)}{l_o^3} d\sigma$$

As compared to Sec. 14, the present collection contains not all solutions given there but only those which are believed to be most significant for the computation of deflections. Only expressions of the deflection component  $\xi$  are given, the corresponding expressions for  $\eta$  being evident. Here A is the point at which  $\xi$  and  $\eta$  are to be computed.

For practical application those solutions that do not contain the inclination  $\tan \beta_1$  or the horizontal derivative  $\partial \Delta g / \partial x$  are preferable for computational reasons given above (a small term should not be computed as the difference of two larger values obtained in different ways). Therefore, D II (reduction to point level) is preferable to D I, and D III' is preferable to D III.

A comparison of the expression (179) for  $\partial \Delta g / \partial h$  with that of  $\bar{G}_1$  given above shows readily that the gradient solution D II is computationally simpler than D III'. In addition,  $G_1$  and  $\bar{G}_1$  depend strongly on the irregularities of topography.

The Pellinen type solution D VIIb, when applicable, shares the computational advantages of D II and D III'. Moreover,  $G'$  is a true function of position just as  $G_1$ , having a unique value at every point of the earth's surface (and being consequently representable by a map or, say, by a set  $5' \times 5'$  mean values), whereas  $(\partial \Delta g / \partial h)(h - h_A)$  and depend, in addition, on the computation point A. On the other hand, the applicability of D I, D II, D III, and D III' is unrestricted (as long as the linear approximation is sufficient), whereas D VIIb presupposes a strong correlation of the free-air anomalies with elevation.

Formulas of a different type will be given in sec. 17.

## 16. SPHERICAL HARMONICS

The conventional formulas for computing coefficients of spherical harmonics of the earth's external gravitational potential require the free-air gravity anomalies to be given on a sphere. As far as the anomalous gravity field is concerned, it is permissible to identify this sphere with the reference ellipsoid -- this is the spherical approximation -- but not with the physical surface of the earth to which the free-air anomalies primarily refer, because the inclination of the terrain is not negligible; cf. sec. 3.

The free-air anomalies at sea level as defined in sec. 9,

$$\Delta g^* = \Delta g - \frac{\partial \Delta g}{\partial h} h,$$

however, are directly suited for computing the spherical-harmonic coefficients by means of the spherical formulas, since they refer to the reference ellipsoid which is represented by a sphere.

Let  $\Delta g_n^*$  be a Laplace surface harmonic so computed from  $\Delta g^*$ , then the harmonic  $T_n$  of the same degree  $n$  of the anomalous potential will be given by

$$T_n = \frac{R}{n-1} \Delta g_n^* . \quad (278)$$

Hence the gradient solution (free-air reduction to sea level) is the most direct method for computing spherical harmonics.

Let us now consider Molodensky's solution (sec. 11). The connection between the anomalous gradient and the Molodensky correction  $G_1$  is according

to (225) given by

$$-\frac{\partial \Delta g}{\partial h} h = G_1 - \frac{R^2}{2\pi} \iint_{\sigma} \frac{h \Delta g - (h \Delta g)_0}{r^3} d\sigma. \quad (279a)$$

According to sec. 1, equations (8) and (9), this formula is equivalent to the following relation between the corresponding harmonics of degree  $n$ :

$$\left( -\frac{\partial \Delta g}{\partial h} h \right)_n = (G_1)_n + \frac{n}{R} (h \Delta g)_n, \quad (279b)$$

so that by (199)

$$\Delta g_n^* = \Delta g_n + n \left( \frac{h}{R} \Delta g \right)_n + (G_1)_n. \quad (280)$$

The sum of the first two terms on the right-hand side is equal to the  $n$ -th degree harmonic of the quantity

$$\Delta g + n \frac{h}{R} \Delta g = \left( 1 + n \frac{h}{R} \right) \Delta g.$$

For lower degrees  $n$  (up to  $n = 5$ , say), the second term is of the order of  $h/R$  and is consequently negligible as a planar approximation. For higher degrees  $n$  this is not true, because of the large factor  $n$ . (Since a factor such as  $n$  always corresponds to some kind of differentiation, this is closely related to the fact that the elevation as such may be negligible, but not so its horizontal derivative, the inclination.)

Hence, for lower-degree harmonics,

$$\Delta g_n^* = (\Delta g + G_1)_n \quad (281)$$

or

$$\left( -\frac{\partial \Delta g}{\partial h} h \right)_n = (G_1)_n. \quad (282)$$

This means that the Molodensky correction can be applied to obtain in a simple way harmonics of lower degree. For higher harmonics (281) does not hold. If we would still wish to apply  $G_1$  in this case, we would have to use the complete expression (280). This expression, however, involves the spherical-harmonic expansion of the product  $h\Delta g$  and is, therefore, impracticable.

The relation (279b) was pointed out by (Molodensky et. al, 1962b).

USE OF TERRAIN CORRECTION. Let us consider solution VIIb of

sec. 14:

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + G') S(\psi) d\sigma - \frac{1}{2} \Delta g \Delta h, \quad (283)$$

where

$$\Delta h = h - h_0, \quad (284)$$

$h_0$  being a mean elevation of the region considered. In exactly the same way as we derived (225) from (222) and (223), we obtain from (283) and (223) the relation

$$G_1 = G' + \frac{R^2}{4\pi} \iint_{\sigma} \frac{\Delta h \Delta g - (\Delta h \Delta g)_p}{r_0^3} d\sigma; \quad (285a)$$

in fact, all we have to do is in (225) to replace  $-h \partial \Delta g / \partial h$  by  $G'$  and  $h \Delta g$  by  $\Delta h \Delta g / 2$ .

This is equivalent to the relation between spherical harmonics

$$(G_1)_n = (G')_n - \frac{n}{2R} (\Delta h \Delta g)_n. \quad (285b)$$

Since  $\Delta h$  is at most of the order of  $h$ , we may to this equation apply the same reasoning by which we deduced (281) from (280), finding that for lower harmonics

$$\Delta g_n^* = (\Delta g + G')_n, \quad (286)$$



because then

$$(G')_n = (G_1)_n = \left( -\frac{\partial \Delta g}{\partial h} h \right)_n. \quad (287)$$

This interesting relation between the lower harmonics (only those!) of three so different quantities was derived by Pellinen (1962, 1964) in different ways.

The main importance of the relation (287) rests in the fact that  $G'$  may in many cases be identified with the conventional terrain correction, according to (262), which can be computed from the topography only, no gravity anomalies being needed. In view of the as yet imperfect global gravity coverage this furnishes a convenient means of estimating the correction (287) to the lower harmonics corresponding to the linear approximation; for a practical estimate see sec. 18.

FIRST-DEGREE HARMONIC. From (278) follows that the first-degree harmonic  $\Delta g_1^*$  must be zero; otherwise the corresponding harmonic  $T_1$  would be infinite. According to (287),

$$\Delta g_1^* = \left( \Delta g - \frac{\partial \Delta g}{\partial h} h \right)_1 = (\Delta g + G_1)_1 + (\Delta g + G')_1 = 0. \quad (288)$$

This condition is satisfied by the gravity anomaly  $\Delta g$ . It may be shown that for any of the integral equation of Part A, except (169), to be solvable a condition equivalent to (288) must be fulfilled.

## 17. THE EXTERNAL GRAVITY FIELD

If the gravity anomalies are given on a level surface, then the external gravity field can be computed by well-known spherical formulas such as (200) (Heiskanen and Moritz, 1967, chap. 6). Since the free-air anomalies  $\Delta g$  refer to the physical surface of the earth, the most natural way is to reduce them to a level surface such as to sea level, to obtain free air anomalies at sea level

$$\Delta g^* = \Delta g - \frac{\partial \Delta g}{\partial h} h \quad (289)$$

according to sec. 9. This expression for  $\Delta g^*$  is valid to a linear approximation; higher-order approximations may be found by an iterative solution of Bjerhammar's integral equation by (174 a, b).

Then the disturbing potential outside the earth is given by (200):

$$T(r_p, \theta, \lambda) = \frac{R^2}{4\pi} \iint_{\sigma} \Delta g^* S(r_p, \psi, R) d\sigma, \quad (290)$$

where

$$r_p = R + h_p,$$

$h_p$  being the elevation above sea level of the point P at which T is to be computed. By (135) and (136) we have

$$S(r_p, \psi, R) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \frac{R^n}{r_p^{n+1}} P_n(\cos \psi); \quad (291)$$

a closed expression is obtained from (136), on replacing  $r$  by  $R$ .

The gravity disturbance vector

$$\vec{\delta} = \text{grad } T \quad (292)$$

is found by forming the partial derivatives of (290) in the usual way (Hirvonen and Moritz, 1963; Heiskanen and Moritz, 1967, chap. 6).

In certain cases, such as for computation of gravity anomalies at flight elevations, it may be of advantage to reduce the surface anomalies  $\Delta g$  to a level surface at a mean elevation of the area under consideration, rather than reducing them to sea level. Reduction to point level is theoretically possible, but less convenient for practical application.

Instead of analytical continuation to some level surface we may also use the Molodensky correction  $G_1$ . In this case  $\Delta g^*$ , eq. (289), is replaced by  $\Delta g + G_1$ , and instead of (290) we have

$$T(r_p, \theta, \lambda) = \frac{R^2}{4\pi} \iint_{\sigma} (\Delta g + G_1) S(r_p, \psi, r) d\sigma, \quad (293)$$

where

$$S(r_p, \psi, r) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \frac{r^n}{r_p^{n+1}} P_n(\cos \psi) \quad (294)$$

in agreement with (135), equation (136) providing a closed expression.

This follows from Brovar's method expressed by equations (135) to (147) of sec. 6. The difference between (147) and (293) is only that (293) suppresses the zero-degree harmonic of  $T$ , which seems to have become customary practice in geodesy, while (147) retains it.

The sum  $\Delta g + G_1$  in (293) is by (137) only a linear approximation to  $\lambda \sec \beta$  (because  $dS = R^2 d\sigma \sec \beta$ ); higher approximations for  $\lambda$  are obtained by an iterative solution of (142) by (143a, b).

Whereas the solution (290) admits of a simple geometrical interpretation, being essentially a spherical formula, such an interpretation is not possible for (293), since

$$r = R + h \quad (295)$$

is not a constant, but a variable depending on the topographic elevation  $h$ . It is the spherical approximation to the radius vector of the earth's physical surface (Fig. 4b). For practical application, the variability of  $r$  constitutes a slight disadvantage of (293) as compared to (290).

As for other methods, the direct application of Molodensky's surface layer (sec. 5) and the use of Green's identities, similar to the way in which the integral equations of sec. 2 have been obtained, leads to complicated and impractical formulas (Brovar, 1963b; Moritz, 1965b).

APPLICATION TO HEIGHT ANOMALIES AND DEFLECTIONS OF THE VERTICAL. The formulas just given for the external gravity field are applicable down to the earth's surface and also on the earth's surface itself. The height anomaly  $\zeta$  is then obtained by dividing the anomalous potential  $T$  by mean gravity  $G$ , and the components  $\xi$ ,  $\eta$  of the deflection of the vertical are obtained as the horizontal derivatives of  $\zeta$ .

Hence (290) and its horizontal derivatives give the formulas of Bjerhammar

$$\zeta = \frac{R^2}{4\pi G} \iint_{\sigma} \Delta g^* S(r_p, \psi, R) d\sigma, \quad (296a)$$

$$\begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \frac{R}{4\pi G} \iint_{\sigma} \Delta g^* \frac{\partial S(r_p, \psi, R)}{\partial \psi} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} d\sigma, \quad (296b)$$

and from (293) we obtain the formulas

$$\zeta = \frac{R^2}{4\pi G} \iint_{\sigma} \lambda \sec \beta S(r_p, \psi, r) d\sigma, \quad (297a)$$

$$\begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \frac{R}{4\pi G} \iint_{\sigma} \lambda \sec \beta \frac{\partial S(r_p, \psi, r)}{\partial \psi} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} d\sigma - \frac{\lambda \cos \beta}{G} \begin{Bmatrix} \tan \beta_1 \\ \tan \beta_2 \end{Bmatrix}. \quad (297b)$$

Equations (296a, b) and (297a, b) are written in such a way that are also valid for higher approximations to  $\Delta g$  and  $\lambda$ , which may be obtained iteratively using (174a, b) and (143a, b), as mentioned above. The quantity  $r$  is expressed in terms of the elevation by (295), and since the computation point  $P$  lies now on the earth's surface, we have

$$r_p = R + h_p, \quad (298)$$

where  $h_p$  is the topographic elevation at  $P$ .

The solution (296a, b) corresponds to free-air reduction to sea level and was given by Bjerhammar (1964); eq. (296a) is identical with eq. (201) of sec. 9.

The solution (297a, b) was published for the zero-order approximation  $\lambda \cos \beta \doteq \Delta g$ ,  $\beta_1 \doteq 0 \doteq \beta_2$  by Hirvonen (1960). Higher approximations to  $\lambda$  were found analytically by Brovar (see sec. 6), of which the linear approximation

$$\lambda \sec \beta = \Delta g + G_1 \quad (299)$$

was used in (293).

The second term on the right-hand side of (297b) is due to the discontinuity of the normal derivative of the function (137) on the physical surface

of the earth  $S$ . As we have seen in sec. 6, this discontinuity along the surface normal has the magnitude  $-\lambda$ . In (140) we have used the vertical component  $-\lambda \cos \beta$  of this discontinuity; now the (negative) horizontal components  $-\lambda \cos \beta \tan \beta_1$  and  $-\lambda \cos \beta \tan \beta_2$  figure in (297b), since

$$\xi = -\frac{1}{G} \frac{\partial T}{\partial x}, \quad \eta = -\frac{1}{G} \frac{\partial T}{\partial y}$$

and, according to (Moritz, 1964, sec. 2.4), the unit normal vector  $\vec{n}$  is given by

$$\vec{n} = (-\cos \beta \tan \beta_1, -\cos \beta \tan \beta_2, \cos \beta).$$

The functions  $S(r_p, \psi, r)$  and  $\partial S(r_p, \psi, r)/\partial \psi$  have been given a computationally convenient form by Hirvonen (1960). He sets

$$t = \frac{r}{r_p} = \frac{R+h}{R+h_p}. \quad (300)$$

Then we have

$$R^2 S(r_p, \psi, r) = r_p t^2 \left[ \frac{2}{D} + 1 - 3D - t \cos \psi \left( 5 + 3 \ln \frac{1+D-t \cos \psi}{2} \right) \right], \quad (301a)$$

$$R \frac{\partial S(r_p, \psi, r)}{\partial \psi} = t^3 \sin \psi \left[ \frac{2}{D^3} + \frac{6}{D} + 3 \frac{D-1+t \cos \psi}{D \sin^2 \psi} - 8 - 3 \ln \frac{1+D-t \cos \psi}{2} \right], \quad (301b)$$

where

$$D^2 = 1 - 2t \cos \psi + t^2.$$

These expressions may also be used in (296a, b) if  $t$  is understood to be  $R/r_p$  instead of (300).

It is evident that the system (296a, b) is considerably simpler than (297a, b). The quantity  $r$  is variable, whereas  $R$  is a constant; above all, however, the expression (297b) for the deflection of the vertical explicitly contains the inclination of the terrain, which is undesirable as we have seen in sec. 15. In fact, the linear approximation of (297b) leads to (269), in the same way as (296b) corresponds to (271) or (272).

Since neither the inclination nor the horizontal derivatives of  $\Delta g$  enter in (296b), this expression shares the accuracy advantage of (272); in addition, (296b) is rigorous in the sense that it is not restricted to the linear approximation. On the other hand, the functions  $S(r_p, \psi, R)$  and  $\partial S(r_p, \psi, R)/\partial \psi$  are more complicated than the simple Stokes and Vening Meinesz functions, so that the main practical significance of (296a, b) is its suitability for higher approximations.

The major disadvantage of the system (297a, b), corresponding to the method of Hirvonen and Brovar, is the inclination term in (297b). This term would be missing if the deflection components  $\xi$  and  $\eta$  were computed outside the earth; the computation point could be as close to the earth's surface  $S$  as we like, as long as it does not coincide with  $S$ . Hence we are tempted to try instead of (297b) the formula

$$\begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = -\frac{R}{4\pi G} \iint_{\sigma} \lambda \sec \beta \frac{\partial S(r_p + \epsilon, \psi, r)}{\partial \psi} \begin{Bmatrix} \cos \alpha \\ \sin \alpha \end{Bmatrix} d\sigma, \quad (302)$$

where  $\epsilon$  is an arbitrarily small, but nonzero, positive number; we could, for

instance, choose  $\epsilon = 1$  cm. This is certainly possible, but unfortunately the simplification, as compared to (297b), is only apparent: if we try to evaluate the effect of the innermost zone in (302), we shall find expressions which contain the inclination after all, and even in much the same way as (297b) does.



## 18. CONCLUSIONS AND RECOMMENDATIONS

in the preceding sections we have tried to draw conclusions as to the suitability of various linear solutions for practical use. We may summarize the results in the following list.

	Gradient Solution	Molodensky Solution
height anomalies	excellent (I b)	excellent (III a)
deflections of vertical	excellent (D II)	good (D III')
spherical { low	excellent	good
harmonics { high	excellent	poor
external field	excellent	good

The designations I b, III a, D II, and D III' refer to the classification of sections 14 and 15. The other solutions of sec. 14 can be ruled out practically because they are less simple, except the Pellinen type solution VII b, which will be considered later.

According to the above table, the best all-purpose method is the gradient solution. As downward continuation, or free-air reduction, to sea level it is ideal for those quantities that are computed on a global scale such as spherical harmonics and height anomalies (or, which is basically the same, geoidal heights). For deflections of the vertical a slight modification, reduction to point level, is appropriate. Here the sea-level anomalies  $\Delta g^*$  are not used directly, which gives to the computation of deflections of the vertical a slightly special character, but this is also in conformity with the fact that deflections are not usually computed

on a global scale. Anyhow, the vertical gradient  $\partial\Delta g/\partial h$  is used for free-air reduction both to sea level and to point level.

Computing this gradient in other ways than (b) is not recommended for general use.

Molodensky's solution III, the correction  $G_1$  being computed by (a), is the most straightforward expression for the height anomaly, although the practical evaluation of  $G_1$  is more problematical than that of  $\partial\Delta g/\partial h$  because  $G_1$  is strongly affected by the irregularities of topography. To get good results for the deflections of the vertical, the modified expression D III' should be used rather than Molodensky's original formula D III. Molodensky's method is of great importance, but on the whole the gradient solution, being simpler and more versatile, appears to be preferable in practice.

The solution VII b, using free-air anomalies combined with a term  $G'$  which is essentially the terrain correction, is fairly well suited for computing height anomalies, deflections of the vertical, and lower-degree spherical harmonics. It is, however, not so universally applicable as the other linear solutions since it presupposes a strong correlation between free-air anomalies and elevation (see sec. 13). The main importance of this solution rests in the fact that it furnishes a link with conventional methods. Since the terrain correction can be computed without needing gravity, estimates of  $G'$  are easily obtained. In addition, we have seen in sections 13 and 15 that the use of  $\Delta g + G'$ , free-air anomalies modified by adding the terrain effect, in Stokes' and Vening Meinesz' integrals without further corrections furnishes height anomalies and deflections of the

vertical at the physical surface of the earth in the sense of Molodensky, at least to a better approximation than if genuine free-air anomalies  $\Delta g$  were used. This result is quite gratifying since in the past Stokes' and Vening Meinesz' integrals have sometimes been evaluated using free-air anomalies conventionally "corrected" by the terrain effect; it is, however, surprising that the results are not geoidal heights and deflections of the vertical at sea level, as one might assume, but height anomalies and deflections at ground level.

This solution is not suited for computing the external gravity field and spherical harmonics of higher degree. Therefore, there is no reason for using it on a larger scale.

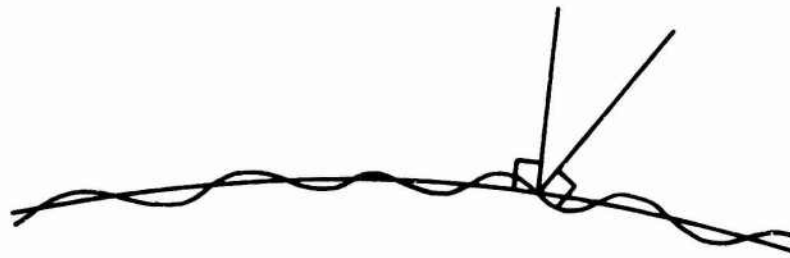
PRACTICAL SIGNIFICANCE OF LINEAR CORRECTIONS. The solutions discussed here consist of spherical formulas such as Stokes' integral, plus small correction terms. The question arises as to the practical significance of these corrections. Studies of mathematical models, such as given in (Arnold, 1960), (Molodenskii et al., 1962a) and (Molodensky et al., 1962b), are useful for understanding extreme situations, but they may tend to give an exaggerated picture. For this reason they should be complemented by test computations in selected areas.

As for height anomalies, Arnold (1960) has obtained a correction of  $-0.2$  m for Mt. Blanc, elevation 4807 m. This small value, which is probably less than the error due to the spherical approximation (sec. 3), seems to indicate that for height anomalies the refinements of Stokes' formula are in general of little

significance, although we shall subsequently see that there is also evidence to the contrary.

In any case, matters are entirely different for deflections of the vertical. It should be carefully kept in mind that 0.2" in the deflection correspond to a linear displacement of 6 meters, and that the effort necessary to obtain the deflections to an accuracy of 0.2" is incomparably greater than that required for the equivalent accuracy of 6 meters in the height anomaly. Fig. 11 indicates that small deformations (corresponding to small corrections to  $\xi$ ) may give rise to considerable changes of inclination (corresponding to corrections to  $\xi$  and  $\eta$ ). Consequently, it is in the case of deflections of the vertical that the refinements of Molodensky's theory are of real practical significance. Except in flat country, the linear corrections to Vening Meinesz' formula are indispensable if an accuracy of  $\pm 0.2''$ , corresponding to an accuracy of  $\pm 10$  m for the absolute position in space, is aimed at. Again, the reader is referred to the examples of (Arnold, 1960).

As for spherical harmonics, especially those of lower degree, it might be expected that the effect of the linear corrections is insignificant, still more so than in the case of the height anomalies. The estimates of Pellinen (1962), however, indicate that this effect is very considerable, of the order of 15 - 20 % of the harmonic coefficients themselves. This surprising result seems to imply that also height anomalies may be noticeably affected by these corrections. This point would probably deserve closer examination.



**Figure II**

ADEQUACY OF THE LINEAR APPROXIMATION. The opposite problem is whether the linear approximation is practically sufficient or whether higher approximations should also be considered. This question, just as the preceding one, is actually beyond the scope of the present report; still, a few arguments may be offered. It all depends on whether the inclination may be considered small or not. An inclination  $\beta$  of  $45^\circ$ , so that  $\tan \beta = 1$ , is obviously not small, but this is certainly an exception. The average inclinations are usually small even in mountainous areas; some kind of averaging is inherent in any topographic map and in any map of gravity anomalies, however dense the gravity survey may be. Still, the fact that the gradient solution is not directly affected by the terrain inclination may be adduced as an argument for preferring it to a solution such as Molodensky's, in which  $G_1$  strongly depends on the irregularities of the terrain.

Certainly, formulas for higher approximations are available. However, the question as to their practical applicability and to the genuineness of the results so obtained comes up. Higher approximations correspond to the use of higher derivatives of the gravity anomaly field and of the topography. It is well known that errors of a given function are increasingly magnified by successive differentiations. An example well-known from geophysical prospecting is the second derivative  $\partial^2 \Delta g / \partial h^2$ , where different methods may yield results that diverge by more than 100 %.

These computational difficulties endanger much more than theoretical difficulties the use of higher approximations. Such theoretical difficulties are, for instance, the impossibility of rigorous downward continuation to sea level and the corresponding mathematical instability of Bjerhammar's integral equation (sec. 7), or question of the analytical convergence of the series expressing the complete solution of Molodensky's integral equation (Molodenskii et al., 1962, pp. 122-3), which apparently has not yet been decided. But over these problems, interesting as they are from a mathematical point of view, we should not forget that we are looking for numerical accuracy, mathematical rigor being only a tool to attain this accuracy. (It is well known from mathematics that analytically divergent, "asymptotic," series yield perfectly useful numerical results!)

In keeping with this, close attention must be paid to the effect of interpolation errors, which are present even with a dense gravity net. In this respect, conventional methods of gravity reduction may have advantages: isostatic and Bouguer anomalies are smoother and easier to interpolate than free-air anomalies. Gravity reduction can indeed be incorporated in the "new theory," but this topic will not be considered here.

RECOMMENDATIONS. Based on the preceding arguments, the gradient solution is suggested for practical application. For computation of height anomalies, spherical harmonics, and the external gravity field, downward continuation to sea level is appropriate, using formulas such as (204) and (290); the sea-level anomaly  $\Delta g^*$  is computed by (199) and (177). For deflections of

the vertical the formula (272), which corresponds to reduction to point level, is suggested. To combine these two different free-air reductions, to sea level and to point level, in a manner suitable for large-scale automatic processing, it is suggested to compute and store;

surface free-air anomaly  $\Delta g$ ,

correction  $-\frac{\partial \Delta g}{\partial h} h$ ,

sea-level free-air anomaly  $\Delta g^*$ ,

anomalous gradient  $\frac{\partial \Delta g}{\partial h}$ ;

the latter being needed for the deflections of the vertical explicitly, that is not only in the form of the correction  $-(\partial \Delta g / \partial h) h$ .

If a higher approximation is desired, the sea-level anomalies  $\Delta g^*$  may be computed by the iterative solution (174 a, b); then the height anomalies and deflections of the vertical are obtained by (296a, b). As a matter of fact, these formulas for  $\zeta$ ,  $\xi$ , and  $\eta$  can also be applied if  $\Delta g^*$  is computed by the linear approximation (199).

These recommendations are based on purely theoretical considerations and are consequently of a somewhat preliminary nature, although they appear plausible. These theoretical considerations should therefore be supplemented by comparative computer studies.



## APPENDIX I

### PLANAR APPROXIMATIONS

PLANAR APPROXIMATION TO STOKES' FUNCTION. As a planar approximation, the first term on the right-hand side of (16) was neglected to get (176). In the same way we may neglect the corresponding terms in equations (10) and (11), whereby they become identical. To this extent, also their inverses (1) and (5) are equivalent, so that the planar approximation to Stokes' function is

$$S(\psi) = \frac{2R}{l_0} . \quad (303)$$

SIMPLIFICATION OF (103). In sec. 4 we have found the solution (103):

$$T = \frac{R}{4\pi} \iint_{\sigma} \left( \Delta g + p_1 + \frac{3q_1}{2R} \right) S(\psi) d\sigma + q_1 . \quad (304)$$

We shall now show that the term  $3q_1/2R$  between the parentheses is negligible as a planar approximation.

According to (97) we may set

$$-q_1 = A + B , \quad (305a)$$

$$-q_2 = A + 2B , \quad (305b)$$

where

$$A = \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0^3} (h - h_p) T d\sigma , \quad (306a)$$

$$B = - \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{l_0^3} \sin \psi \frac{\partial h}{\partial \psi} T d\sigma . \quad (306b)$$

Let us first consider the term A. Its effect on T is

$$\delta T_A = \frac{R}{4\pi} \iint_{\sigma} \left( \frac{3A}{2R} \right) S(\psi) d\sigma. \quad (307)$$

If we compare A with  $G_1$ , eq. (226), we see that the only difference is that  $\Delta g$  in  $G_1$  is replaced by T in A. By comparing (222) and (223) we find

$$\frac{R}{4\pi} \iint_{\sigma} G_1 S(\psi) d\sigma = - \frac{R}{4\pi} \iint_{\sigma} \frac{\partial \Delta g}{\partial h} h S(\psi) d\sigma - h \Delta g;$$

whence, on replacing  $\Delta g$  by T,

$$\frac{R}{4\pi} \iint_{\sigma} A S(\psi) d\sigma = - \frac{R}{4\pi} \iint_{\sigma} \frac{\partial T}{\partial h} h S(\psi) d\sigma - h T,$$

so that (307) becomes

$$\delta T_A = \frac{R}{4\pi} \iint_{\sigma} \frac{3h}{2R} \left( \Delta g + \frac{2T}{R} \right) S(\psi) d\sigma - \frac{3h}{2R} T. \quad (308)$$

Now the integrand is of the order of  $(h/R) \Delta g$  which is negligible with respect to the gravity anomaly  $\Delta g$ , which is the main term of the integrand of (304), and the last term of (308) is of the order of  $(h/R) T$ , which is negligible with respect to T. Hence the complete effect  $\delta T_A$  is negligible as a planar approximation.

As for B, the comparison of (306a) and (306b) shows that A and B are of the same magnitude since  $\sin \psi \partial h / \partial \psi$  is of the same order of magnitude as  $h - h_p$  for small  $\ell_0$ . Therefore we may conclude that the effect  $\delta T_B$  will be of the same order of magnitude as  $\delta T_A$  and will consequently be also negligible as a planar approximation.

The weak spot in this argument is that the above-mentioned equality of orders of magnitude of  $\sin \psi \partial h / \partial \psi$  and  $h - h_p$  holds for small  $\ell_0$  (of the order of a few kilometers) only. It is therefore desirable to have an independent check, although the computations will be somewhat lengthy.

We consider the effect of  $q_1$ , which by (305a) is the combined effect of A and B:

$$\delta T_c = \delta T_A + \delta T_B ; \quad (309)$$

we shall set

$$C = -q_1 - \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_c^3} \left( h - h_p - \sin \psi \frac{\partial h}{\partial \psi} \right) T d\sigma . \quad (310)$$

By (220) this is equivalent to

$$C = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{\ell_c^3} (h - h_p) \frac{\partial T}{\partial \psi} d\sigma \quad (310')$$

or

$$C = D + E \quad (311)$$

where, by (38),

$$D = -h_p \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{\ell_c^3} \frac{\partial T}{\partial \psi} d\sigma = h \frac{R^2}{2\pi} \iint_{\sigma} \bar{D} \left( T, \frac{1}{\ell_o} \right) d\sigma , \quad (312a)$$

$$E = \frac{R^2}{2\pi} \iint_{\sigma} \frac{\sin \psi}{\ell_c^3} h \frac{\partial T}{\partial \psi} d\sigma . \quad (312b)$$

The transformation of (312a) by means of (33) yields

$$D = -h \frac{R^2}{2\pi} \iint_{\sigma} \frac{\Delta_2 T}{\ell_o} d\sigma .$$

Since  $\Delta_2 T = \partial \Delta g / \partial h$  by (180), this becomes

$$D = -h \frac{R^2}{2\pi} \iint_{\sigma} \frac{1}{\ell_o} \left( \frac{\partial \Delta g}{\partial h} \right) d\sigma ;$$

and since this integral is the planar inverse of (176), the expression for D

reduces to

$$D = h \Delta g ; \quad (313)$$

hence the effect on  $T$  is in analogy to (307) given by

$$\delta T_0 = \frac{R}{4\pi} \iint \frac{3h}{2R} \Delta g S(\psi) d\sigma. \quad (314)$$

As for (312b), the effect of  $E$  is

$$\begin{aligned} \delta T_c &= \frac{R}{4\pi} \iint_{\sigma} \frac{3E}{2R} S(\psi) d\sigma \\ &= \frac{3}{2R} \cdot \frac{R}{4\pi} \iint_{\sigma} E S(\psi) d\sigma. \end{aligned}$$

Now (312b) corresponds to (231),  $h$  replacing  $\Delta g$ , and  $T$  replacing  $h$ . Hence the integral

$$\frac{R}{4\pi} \iint_{\sigma} E S(\psi) d\sigma,$$

which corresponds to (232), is by (240) equal to

$$\frac{R}{4\pi} \iint_{\sigma} h \frac{\partial T}{R \partial \psi} S(\psi) d\sigma,$$

so that the effect of  $E$  becomes

$$\delta T_c = \frac{R}{4\pi} \iint_{\sigma} \frac{3h}{2R} \frac{\partial T}{R \partial \psi} S(\psi) d\sigma. \quad (315)$$

Adding (314) and (315) according to (311) we find the effect of  $C = -q_1$  to be

$$\delta T_c = \frac{R}{4\pi} \iint_{\sigma} \frac{3h}{2R} \left( \Delta g + \frac{\partial T}{R \partial \psi} \right) S(\psi) d\sigma. \quad (316)$$

Since all components of the "gravity disturbance vector"  $\text{grad } T$  will be of the same order of magnitude, the horizontal component

$$\frac{\partial T}{R \partial \psi}$$

will be of the order of the vertical component

$$- \frac{\partial T}{\partial r} = \Delta g + \frac{2T}{R}$$

and consequently of  $\Delta g$  itself. Hence the effect of C, eq. (316), is negligible as a planar approximation for the same reason as (308) is.

As a by-product we find the effect of B by subtracting (308) from (316) to be

$$\delta T_B = \frac{R}{4\pi} \iint_{\sigma} \frac{3h}{2R} \left( \frac{\partial T}{R \partial \psi} - \frac{2T}{R} \right) S(\psi) d\sigma + \frac{3h}{2R} T, \quad (317)$$

whose planar approximation is likewise zero.

Hence (304) reduces in fact to

$$T = \frac{R}{4\pi} \iint_{\sigma} (\Delta g + p_1) S(\psi) d\sigma + q_1, \quad (318)$$

which is identical with (104).

## APPENDIX II

### MEAN CURVATURE

We shall now derive the formula (191) for the difference of mean curvature between two surfaces  $S$  and  $\bar{S}$ , whose distance, measured along the normal to  $S$ , is  $\zeta$ .

A formula for a similar purpose is eq. (8) of (Weatherburn, 1930, p. 173); however, it is laborious to adapt it to our present case. Therefore we shall proceed in a more direct manner.

Denote the position vector of  $S$  by  $\vec{x}$  and its normal vector by  $\vec{n}$ . Then the relation

$$\Delta_2 \vec{x} = -2J\vec{n} \quad (319)$$

holds, where  $\Delta_2$  is the surface Laplacian defined in sec. 1. This formula is equivalent to expressions given in (Weatherburn, 1927, p. 231) and (McConnell, 1931, p. 203, example 3), only the notations for mean curvature are different. Our  $J$  corresponds to the negative of McConnell's  $H$  and of Weatherburn's  $J/2$ .

We shall use a tensor notation equivalent to McConnell's, except that we are using ordinary instead of Greek indices for surface tensors, and a notation such as  $\vec{x}$ , without any indices, for space vectors. Then (319) may be written

$$2J\vec{n} = -a^{ij} \vec{x}_{i,j} = -a^{ij} \left( \vec{x}_{i,j} - \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} \vec{x}_k \right), \quad (320)$$

where

$$a_{ij} = \vec{x}_i \cdot \vec{x}_j \quad (321a)$$

is the first fundamental tensor of the surface, and  $\vec{x}_i$  and  $\vec{x}_{,j}$  denote partial derivatives with respect to the surface coordinates. The second and third fundamental tensors are, as usually,

$$b_{ij} = -\vec{x}_i \cdot \vec{n}_j = \vec{x}_{,i} \cdot \vec{n} \quad (321b)$$

$$c_{ij} = \vec{n}_i \cdot \vec{n}_j = -\vec{n}_{,i} \cdot \vec{n} \quad (321c)$$

with the relation

$$c_{ij} = a^{kl} b_{ik} b_{jl} \quad (322)$$

connecting the three fundamental tensors. If  $K$  is the Gaussian curvature of the surface, then

$$a^{ij} c_{ij} = b^{ij} b_{ij} = 4J^2 - 2K \quad (323)$$

These relations may be found in any textbook on tensor analysis, such as (McConnell, 1931).

The position vector  $\vec{x}$  of  $\bar{S}$  is evidently given by

$$\vec{x} = \vec{x} + \zeta \vec{n} \quad (324)$$

By (324) and (321 a) we have for the first fundamental tensor of  $\bar{S}$

$$\bar{a}_{ij} = a_{ij} - 2\zeta b_{ij} \quad (325)$$

retaining only terms linear in  $\zeta$ . The contravariant fundamental tensor is given by

$$\bar{a}^{ij} = a^{ij} + 2\zeta b^{ij} \quad (326)$$

since by (325) and (326)

$$\bar{a}_{i,j} \bar{a}^{jk} = a_{i,j} a^{jk} = \delta_i^k.$$

We further set

$$\overline{\left\{ \begin{smallmatrix} k \\ i j \end{smallmatrix} \right\}} = \left\{ \begin{smallmatrix} k \\ i j \end{smallmatrix} \right\} + r_{ij}^k, \quad (327)$$

where we need not evaluate the linear residuals  $r_{ij}^k$ .

Multiplying (320) by  $\vec{n}$  and taking (321b) into account we find

$$2J = -a^{ij} b_{i,j}. \quad (328)$$

Let us now multiply the equation corresponding to (320),

$$2\bar{J}\vec{n} = -\bar{a}^{ij} \vec{x}_{i,j},$$

by  $\vec{n}$ . Then, since as a linear approximation

$$\vec{n} \cdot \vec{n} = \cos \beta \doteq 1,$$

we have

$$2\bar{J} = -\bar{a}^{ij} \vec{x}_{i,j} \cdot \vec{n}. \quad (329)$$

By differentiating (324) we find

$$\begin{aligned} \vec{x}_{i,j} &= \vec{x}_{i,j} - \left\{ \begin{smallmatrix} k \\ i j \end{smallmatrix} \right\} \vec{x}_k \\ &= \vec{x}_{i,j} + \zeta_{i,j} \vec{n} + \zeta_i \vec{n}_j + \zeta_j \vec{n}_i + \zeta \vec{n}_{i,j} \\ &\quad \left\{ \begin{smallmatrix} k \\ i j \end{smallmatrix} \right\} (\vec{x}_k + \zeta_k \vec{n} + \zeta \vec{n}_k) - \Delta_{ij}^k x_k, \end{aligned}$$

so that

$$\begin{aligned} \vec{x}_{i,j} \cdot \vec{n} &= b_{i,j} + \zeta_{i,j} - \zeta c_{i,j} - \left\{ \begin{smallmatrix} k \\ i j \end{smallmatrix} \right\} \zeta_k \\ &= b_{i,j} + \zeta_{i,j} - \zeta c_{i,j}. \end{aligned}$$



By means of this equation and of (326) we obtain from (329)

$$\begin{aligned} 2\bar{J} &= - (a^{i,j} + 2\zeta b^{i,j}) (b_{i,j} + \zeta_{i,j} - \zeta c_{i,j}) \\ &= 2J - 2\zeta b^{i,j} b_{i,j} - a^{i,j} \zeta_{i,j} + \zeta a^{i,j} c_{i,j}, \end{aligned}$$

so that

$$\bar{J} = J - (2J^2 - K) \zeta - \frac{1}{2} \Delta_g \zeta. \quad (330)$$

Thus the derivation of (191) is finished.

This equation, without derivation, was given in (Moritz, 1962).

A similar formula was published by Marussi (1957).

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